Geometric methods for inverse Galois theory
ガロアの逆問題における幾何学的方法

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Outline

1. The Inverse Galois Problem, G-covers and Hurwitz schemes
2. Rings of components of Hurwitz schemes and their geometry
3. Fields of definition of components of Hurwitz schemes

Part 1:
The Inverse Galois Problem, G-covers and Hurwitz schemes

The Inverse Galois Problem

Question (Inverse Galois Problem (IGP))

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Remark (Van der Waerden conjecture, proposed proof by Barghava '21)

Among the $(2H+1)^n$ unitary polynomials of degree n whose coefficients are in $\{-H, \dots, H\}$, only $O(H^{n-1})$ have a Galois group not isomorphic to $\mathfrak{S}_n$.

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The Malle conjecture: a stronger conjecture which predicts the exact distribution of field extensions.

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G finite group.

Question (Regular Inverse Galois Problem (RIGP) for G)

Is there a **regular** *Galois extension $F \mid \mathbb{Q}(t)$ of Galois group G?*

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Why bother?

- RIGP $\Rightarrow$ IGP.
  Follows from **Hilbert's Irreducibility Theorem** $\rightsquigarrow$ Basis of modern Inverse Galois Theory.

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- Extensions of function fields have a geometric meaning:

$$\left\{ \begin{matrix} \text{regular Galois extension of } \mathbb{Q}(t) \\ \text{of Galois group } G \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{connected } G\text{-cover} \\ \text{defined over } \mathbb{Q}. \end{matrix} \right\}.$$

  What are these objects?

G-covers

Conf_n: configuration space for n (distinct, unordered) points in $\mathbb{P}^1(\mathbb{C})$.

Definition

A G-**cover** branched at $\underline{t} \in \mathrm{Conf}_n$ is an unramified cover p of $\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$, equipped with a morphism $\alpha : G \to \mathrm{Aut}(p)$ which induces a free transitive action on every fiber.
A **marked** G-**cover** also comes with a marked point above a basepoint $t_0 \in \mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$.

Another perspective: a dominant finite morphism from a smooth curve Y onto $\mathbb{P}^1_{\mathbb{C}}$, étale outside $\underline{t}$ + an action of G, free/transitive on every unramified fiber.

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If the curve Y is irreducible, its function field is a Galois extension of $\mathbb{C}(t)$ of group G:

$$\{\text{Irreducible } G\text{-covers}\} \simeq \left\{ \begin{array}{c} \text{Galois extensions} \\ F \mid \mathbb{C}(t) \text{ of group } G \end{array} \right\}$$

Fields of definition

- A G-cover Y is *defined over* $\mathbb{Q}$ if there is a G-cover $Y' \to \mathbb{P}^1_{\mathbb{Q}}$ such that the following diagram is cartesian:

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ \mathbb{P}^1_{\mathbb{C}} & \longrightarrow & \mathbb{P}^1_{\mathbb{Q}} \end{array}$$

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- **Main takeaway:** G-covers defined over $\mathbb{Q}$ (with Y' irreducible) correspond to regular Galois extensions of $\mathbb{Q}(t)$ of group G.

Question (Geometrical reformulation of RIGP for G)

Is there a G-cover defined over $\mathbb{Q}$?

Hurwitz schemes

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- **Thompson '84 :** IGP for the Monster group *(rigidity methods)*. Later reinterpreted as the fact that there is an irreducible component X of $\mathrm{Hur}(M, 3)$ such that $X \to \mathrm{Conf}_3$ is an isomorphism.

Hurwitz spaces and RIGP

Moduli space property: if S is a $\mathbb{Q}$-scheme, then there is a (natural) bijection between:

- Morphisms $S \to \mathrm{Hur}^*(G, n)$
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Take $S = \mathrm{Spec}\,\mathbb{Q}$:

$$\{\mathbb{Q}\text{-points of } \mathrm{Hur}^*(G, n)\} \simeq \left\{ \begin{array}{c} \text{marked } G\text{-covers} \\ \text{defined over } \mathbb{Q} \\ \text{with } n \text{ branch points} \end{array} \right\}$$

Remark: issues due to the fact that in general the Hurwitz moduli scheme $\mathrm{Hur}(G, n)$ for *non-marked* G-covers is a *coarse* moduli space. When G is centerfree, it is a fine moduli space. In this case its $\mathbb{Q}$-points indeed correspond to non-marked covers defined over $\mathbb{Q}$.

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The regular inverse Galois problem for G becomes a problem of diophantine geometry:

Question

Does the Hurwitz scheme $\mathrm{Hur}(G, n)$ have rational points, for some $n \in \mathbb{N}$?

Use of geometrical methods

Ellenberg, Venkatesh, Westerland, Tran '16-'17, their strategy:

- Homological information about Hurwitz spaces (combinatorial methods)
- $\rightsquigarrow$ Count $\mathbb{F}_q$-points using Grothendieck-Lefschetz methods (i.e. extensions of $\mathbb{F}_q(t)$)
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Two reasons to study irreducible components of Hurwitz schemes:

- Homology of Hurwitz spaces (including H_0) is central in the strategy above
- A $\mathbb{Q}$-point has to belong to a component defined over $\mathbb{Q}$

Part 2:
Rings of components of Hurwitz schemes and their geometry

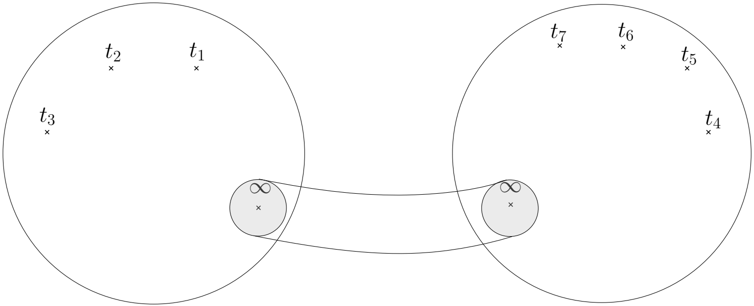
Gluing and patching

Gluing operation on marked G-covers:

$$\begin{pmatrix} G\text{-cover} \\ \text{monodromy group } H_1 \\ n \text{ branch points} \end{pmatrix} \times \begin{pmatrix} G\text{-cover} \\ \text{monodromy group } H_2 \\ n' \text{ branch points} \end{pmatrix} = \begin{pmatrix} G\text{-cover} \\ \text{monodromy group } \langle H_1, H_2 \rangle \\ n + n' \text{ branch points} \end{pmatrix}$$

Glue two projective lines together

$\Rightarrow$ get a single projective line with more branch points!



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Patching theory (Harbater): A similar operation can be done with G-covers defined over K when K is a *complete* valued field $(\mathbb{C}, K((t)), \mathbb{Q}_p, \ldots)$.

The ring of components

k a field of characteristic zero.

Definition (Ring of components)

The ring of components $R(G)$ is the graded k-algebra $\bigoplus_n H_0(\mathrm{Hur}^*(G,n),k)$ equipped with the multiplication induced by gluing.

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For $\mathbb{P}^1$:

Theorem (S. 22)

$R(G)$ is a commutative graded k-algebra of finite type.

$\rightsquigarrow$ I can define the scheme $\mathrm{Proj}\, R(G)$ (variant of Spec for graded rings) and study it

The splitting number

- c = conjugacy class of G;
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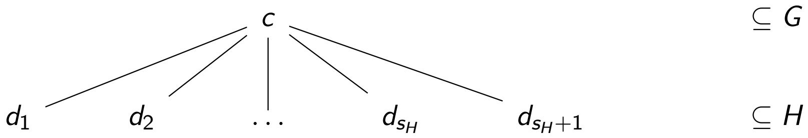

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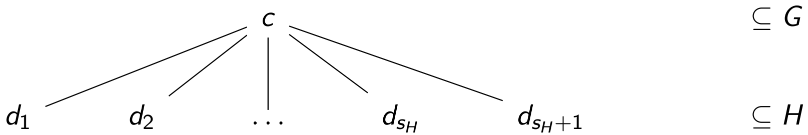

- EVW prove homological stability when $s_H = 0$ $\rightsquigarrow$ how does it generalize?

Geometry of rings of components

Theorem (S. 22)

The Krull dimension d of $\mathrm{Proj}\, R(G, c)$ equals $\max_{H \subseteq G} s_H$, and the count of components with n branch points grows like n^d.

I have a more precise version, dealing with each subgroup H.

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I have a more precise version, dealing with each subgroup H.

I also have an expression of the leading coefficient:

Theorem (S. 22)

The count of components with n branch points and monodromy group H has an average order given by:

$$\frac{|H_2(H, c \cap H)|}{|H^{ab}|\, s_H!} n^{s_H}$$

for an explicit quotient $H_2(H, c \cap H)$ of the second group homology of H.

Example: symmetric groups

$G = \mathfrak{S}_d$ and c the conjugacy class of transpositions.

- The ring of components admits the presentation:

$$R(\mathfrak{S}_d, c) = \frac{k[(X_{ij})_{1 \leq i < j \leq d}]}{(X_{ij}X_{jk} = X_{ik}X_{jk} = X_{ij}X_{ik})_{1 \leq i < j < k \leq d}}$$

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- Description of $\mathrm{Proj}\, R(\mathfrak{S}_d, c)(k)$ as a subvariety of $\mathbb{P}^{\frac{d(d-1)}{2}-1}(k)$ of dimension $\lfloor d/2 \rfloor - 1$:
  - one vertex e_A for each subset $A \subseteq \{1, \dots, d\}$ of size ≥ 2 $\rightsquigarrow \mathfrak{S}_A$
  - the line (e_A, e_B) when A, B are disjoint $\rightsquigarrow \mathfrak{S}_A \times \mathfrak{S}_B$
  - the plane (e_A, e_B, e_C) when A, B, C are disjoint $\rightsquigarrow \mathfrak{S}_A \times \mathfrak{S}_B \times \mathfrak{S}_C$
  - etc.

Example of $\mathfrak{S}_3$

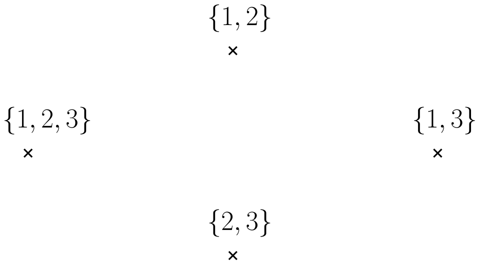


Figure: $\mathrm{Proj}\, R(\mathfrak{S}_3, c)$

dimension 0 $\rightsquigarrow$ situation of EVW (homological stability)

Example of $\mathfrak{S}_4$

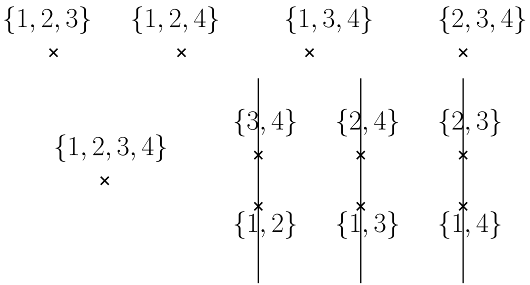
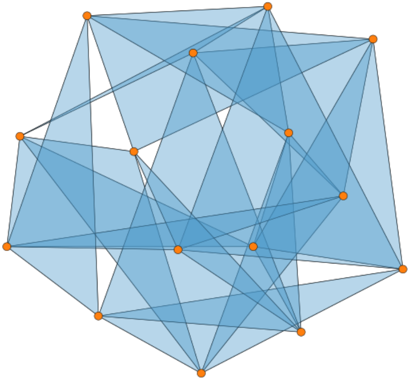


Figure: Proj $R(\mathfrak{S}_4, c)$

dimension 1 $\rightsquigarrow$ no homological stability (linear growth)
(schematic drawing: the actual drawing is in 5D...)

Example of $\mathfrak{S}_6$?

To observe dimension 2, the smallest example is $d = 6$. Problem: many irreducible components to draw (77 vertices, 160 lines, 15 planes) en 14D.
I draw only the part of the Proj corresponding to subsets of $\{1, 2, 3, 4, 5, 6\}$ of size 2, i.e. the 15 planes, represented as triangles:



Part 3:
Fields of definition of components of Hurwitz schemes

Fields of definition and products

How does the product of components behave with the fields of definition?
(*field of definition* = of the underlying component of non-marked covers)
If x, y are components defined over $\mathbb{Q}$, is xy also defined over $\mathbb{Q}$?

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(*field of definition* = of the underlying component of non-marked covers)
If x, y are components defined over $\mathbb{Q}$, is xy also defined over $\mathbb{Q}$?
Probably not true in general. A partial answer:

Theorem (Cau 12)

If x, y are components defined over $\mathbb{Q}$,

$$\left\{ x^g y^{g'} \,\middle|\, g, g' \in G \right\} \text{ is stable under the action of } \mathrm{Gal}\left(\overline{\mathbb{Q}} \mid \mathbb{Q}\right).$$

If this set is a singleton $\Rightarrow$ xy is defined over $\mathbb{Q}$.

A well-behaved situation

Theorem (S. 23)

If x, y are components defined over $\mathbb{Q}$, denote their respective monodromy groups by H_1, H_2. If $H_1 H_2 = \langle H_1, H_2 \rangle$ then for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$:

$$\sigma.(xy) = (\sigma.x)(\sigma.y).$$

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Corollary

If x, y are components defined over $\mathbb{Q}$ and their monodromy groups H_1, H_2 satisfy $H_1 H_2 = G$, then xy is defined over $\mathbb{Q}$.

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Corollary

If x is a component defined over $\mathbb{Q}$ and $n \geq 1$, then x^n is defined over $\mathbb{Q}$.

Lifting invariant methods

The lifting invariant is an invariant (introduced by EVW) with values in a group.
It can be used to study fields of definition. An example:

Theorem (S. 23)

For a constant M depending only on the group G, if x, y are components defined over $\mathbb{Q}$ and xy has G as its monodromy group, then $(xy)^M$ is defined over $\mathbb{Q}$.

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Ingredients for the proof:

- The lifting invariant of $x^\gamma y^{\gamma'}$ is equal to that of xy (not true for covers of $\mathbb{A}^1$!)
- If every conjugacy class of G is the conjugacy class of either 0 or $\geq M$ local monodromy elements, then the component is entirely determined by its lifting invariant (generalization of the Conway-Parker theorem)
- This implies $x^\gamma y^{\gamma'} = xy$. Conclude by Cau's theorem.

Patching methods

Another theorem, proved using Harbater's patching method:

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- See f_n, g_n as covers defined over the complete field $K_n((t))$ and glue them together into a cover h_n defined over $K_n((t))$, which is "in" the component $x^{\gamma_n} y^{\gamma'_n}$ for some $\gamma_n, \gamma'_n \in G$.

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If x, y are components defined over $\mathbb{Q}$, there are $\gamma, \gamma' \in G$ such that $x^\gamma y^{\gamma'}$ is defined over $\mathbb{Q}$.

Sketch of proof.

- Using Hilbert's irreducibility theorem, construct an infinite sequence of fields $K_1, K_2, \ldots$, linearly disjoint over $\mathbb{Q}$, such that there are covers $f_n \in x, g_n \in y$ defined over K_n.
- See f_n, g_n as covers defined over the complete field $K_n((t))$ and glue them together into a cover h_n defined over $K_n((t))$, which is "in" the component $x^{\gamma_n} y^{\gamma'_n}$ for some $\gamma_n, \gamma'_n \in G$.
- Since there are finitely many components of the form $x^\gamma y^{\gamma'}$, at least two of the covers $h_n, h_{n'}$ belong to the same component $x^\gamma y^{\gamma'}$.

Patching methods

Another theorem, proved using Harbater's patching method:

Theorem (S. 23)

If x, y are components defined over $\mathbb{Q}$, there are $\gamma, \gamma' \in G$ such that $x^{\gamma} y^{\gamma'}$ is defined over $\mathbb{Q}$.

Sketch of proof.

- Using Hilbert's irreducibility theorem, construct an infinite sequence of fields $K_1, K_2, \ldots$, linearly disjoint over $\mathbb{Q}$, such that there are covers $f_n \in x, g_n \in y$ defined over K_n.
- See f_n, g_n as covers defined over the complete field $K_n((t))$ and glue them together into a cover h_n defined over $K_n((t))$, which is "in" the component $x^{\gamma_n} y^{\gamma'_n}$ for some $\gamma_n, \gamma'_n \in G$.
- Since there are finitely many components of the form $x^{\gamma} y^{\gamma'}$, at least two of the covers $h_n, h_{n'}$ belong to the same component $x^{\gamma} y^{\gamma'}$.
- The field of definition of $x^{\gamma} y^{\gamma'}$ is included in:

$$\overline{\mathbb{Q}} \cap K_n((t)) \cap K_{n'}((t)) = \mathbb{Q}.$$

Publications

My preprint: "The Geometry of Rings of Components of Hurwitz Spaces". **arXiv:2210.12793**
Forthcoming: "Fields of Definition of Components of Hurwitz Spaces".