

COUNTING TWO-STEP NILPOTENT WILDLY RAMIFIED EXTENSIONS OF FUNCTION FIELDS

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ABSTRACT. We study the asymptotic distribution of wildly ramified extensions of function fields in characteristic $p > 2$, focusing on (certain) p -groups of nilpotency class at most 2. Rather than the discriminant, we count extensions according to an invariant describing the last jump in the ramification filtration at each place. We prove a local–global principle relating the distribution of extensions over global function fields to their distribution over local fields, leading to an asymptotic formula for the number of extensions with a given global last-jump invariant. A key ingredient is Abrashkin’s nilpotent Artin–Schreier theory, which lets us parametrize extensions and obtain bounds on the ramification of local extensions by estimating the number of solutions to certain polynomial equations over finite fields.

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1. INTRODUCTION

For the whole article, we fix a prime number p . If F is a field, we denote by $\Gamma_F := \text{Gal}(F^{\text{sep}}|F)$ its absolute Galois group.

1.1. **Context.** The asymptotic distribution of field extensions (usually counted by discriminant) is an actively studied topic. Over a number field F , the case of abelian extensions was solved in [Wri89] using the description of $\text{Gal}(F^{\text{ab}}|F)$ given by class field theory, and the case of extensions of degree ≤ 5 was solved in [DH69, Bha05, Bha10, BSW15] using explicit parametrizations. Significant progress has also been made for nilpotent extensions in [KM04, KP23]. Although the general problem remains wide open, precise conjectures predict the expected distribution of class groups and Galois groups [CL84, Mal02]. Over function fields, similar conjectures have been made when considering only tamely ramified extensions. Results consistent with these expectations were obtained recently in [EVW16, ETW23, LL25] by counting \mathbb{F}_q -points of moduli spaces of tamely ramified covers of the line (Hurwitz spaces).

Meanwhile, the distribution of wildly ramified extensions of function fields of characteristic p (both local and global) is much more mysterious, and there is not even a conjecture (see however [DY25, Conjecture 6.10]). For abelian p -extensions, asymptotics have been described in [Lag12, Lag15, KM20, Pot26, Gun26]. For non-abelian p -extensions, very little is known: only the local distribution of extensions whose Galois group is a certain generalization of Heisenberg groups (different from the generalized Heisenberg groups which we consider in Section 6) has been

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described in [Mü22]. Constructions of moduli spaces for wildly ramified covers of curves, as in the articles [Har80, BM00, FM02, Pri02, Zha19, TY23, DH24], highlight how different these spaces are from usual Hurwitz spaces, making it unlikely that the strategy of [ETW23, LL25] can be straightforwardly adapted.

The goal of this article is to study the distribution of extensions of function fields of characteristic $p > 2$ for the p -groups arising as central extensions of abelian groups, i.e., p -groups of nilpotency class ≤ 2 .

1.2. The last jump. In this paper, we do not count using the discriminant, but we use an invariant obtained by describing the last jump in the ramification filtration at each place, in the upper numbering. Let G be a finite group (seen as a discrete topological group). If \mathfrak{F} is a local field and K is an (étale) G -extension of \mathfrak{F} , corresponding to a continuous group homomorphism $\gamma: \Gamma_{\mathfrak{F}} \rightarrow G$ (see Subsection 2.1), we define

$$\text{lastjump}(K|\mathfrak{F}) := \inf \{v \in \mathbb{R}_{\geq 0} \mid \gamma(\Gamma_{\mathfrak{F}}^v) = 1\}$$

where $\Gamma_{\mathfrak{F}}^v$ denotes the v -th ramification subgroup of the absolute Galois group of \mathfrak{F} , in the upper numbering (see [Ser62, Chap. IV, §3]). When the choice of the local field \mathfrak{F} is implied by the context, we define $\text{lastjump}(K) := \text{lastjump}(K|\mathfrak{F})$.

Now, let K be an (étale) G -extension of a global field F . At each place P of F , we define $\text{lastjump}_P(K) := \text{lastjump}(K \otimes_F F_P | F_P)$, where F_P is the completion of F at P . These local invariants are then assembled into the following global invariant, mimicking the way invariants like the discriminant (or rather its degree) behave:

$$\text{lastjump}(K) := \sum_P \deg(P) \cdot \text{lastjump}_P(K). \quad (1.1)$$

In its principle, this invariant is more closely related to the “product of the ramified primes” (used for example in [Woo10]) than to the discriminant, but it adds weights to the primes depending on how wild the ramification at each prime is. Note however that our invariant does not distinguish unramified primes from tamely ramified primes.

When G is a p -group and F has characteristic p , tame ramification is impossible, so that $\text{lastjump}(K)$ (resp. $\text{lastjump}_P(K)$) vanishes if and only if K is an unramified extension (resp. unramified at P).

The invariants $\text{lastjump}_P(K)$ and $\text{lastjump}(K)$ are rational numbers whose denominators divide $|G|$. When G is an abelian group, the last jump is always an integer by the Hasse–Arf theorem, and it coincides with the exponent of the “Artin–Schreier conductor”, an invariant for which asymptotics were given in [Gun26]. As discussed there, counting by Artin–Schreier conductor gives simpler, more uniform results than counting by discriminant.

1.3. Main results. Assume now that $p > 2$, and let G be a non-trivial finite p -group of nilpotency class at most 2, i.e., such that $[G, G] \subseteq Z(G)$. As we explain in Lemma 2.14, the p -torsion elements of G form a group, which we denote by $G[p]$. Let $F := \mathbb{F}_q(T)$ be a rational (global) function field of characteristic p , and let $\text{ÉtExt}(G, F)$ be the set of isomorphism classes of (étale) G -extensions of F (see Subsection 2.1). Our first main result is the following exact local–global principle:

Theorem 1.1 (cf. Theorem 5.2). *For every place P of $F = \mathbb{F}_q(T)$, let $N_P \in \mathbb{Q}_{\geq 0}$. Assume that $N_P = 0$ for all but finitely many places. Then,*

$$\sum_{\substack{K \in \text{ÉtExt}(G, F): \\ \forall P, \text{lastjump}_P(K) = N_P}} \frac{1}{|\text{Aut}(K)|} = \prod_P \sum_{\substack{K_P \in \text{ÉtExt}(G, F_P): \\ \text{lastjump}(K_P) = N_P}} \frac{1}{|\text{Aut}(K_P)|}.$$

This local–global principle allows us to determine the asymptotics of G -extensions of $\mathbb{F}_q(T)$ using estimates for the number of extensions of the local fields $\mathbb{F}_{q^d}((T))$ for $d \geq 1$. The following global asymptotics are the main results of this article.

Theorem 1.2 (cf. Theorem 5.6). *Let*

$$r := \log_p |G[p]| \quad \text{and} \quad M := \begin{cases} 1 & \text{if } G[p] \text{ is abelian,} \\ 1 + p^{-1} & \text{otherwise.} \end{cases}$$

If $G[p]$ is non-abelian, assume that $|G[p]| \leq p^{p-1}$. Assume moreover that q is a large enough power of p (depending on the group G). Then, there is a function $C : \mathbb{Q}/M\mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ with $C(0) \neq 0$, such that for rational $N \rightarrow \infty$, we have

$$\sum_{\substack{K \in \text{ÉtExt}(G,F): \\ \text{lastjump}(K)=N}} \frac{1}{|\text{Aut}(K)|} = C(N \bmod M) \cdot q^{\frac{r+1}{M} \cdot N} + o\left(q^{\frac{r+1}{M} \cdot N}\right).$$

The hypothesis that q is large is needed due to a technical limitation of our local counting methods. An explicit lower bound on q can be deduced from the proof of Theorem 5.6, but we presume that the conclusion may hold for all q . We show that the conclusion indeed holds for all q if G has exponent p , or more generally if G satisfies the hypothesis of Proposition 4.15.

For concrete groups G , it can be possible to overcome the restriction that $|G[p]| \leq p^{p-1}$. As an illustration, we carry out the necessary computations for generalized Heisenberg groups:

Theorem 1.3 (cf. Theorem 6.7). *Let $k \geq 1$, and let $G := H_k(\mathbb{F}_p)$ be the generalized Heisenberg group defined in Equation (6.4). There are explicit constants $A \in \mathbb{Q}_{>0}, B \in \mathbb{N}_{>0}, M \in \mathbb{Q}_{>0}$ (cf. Subsection 6.6) and a function $C : \mathbb{Q}/M\mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ with $C(0) \neq 0$, such that for rational $N \rightarrow \infty$, we have*

$$\sum_{\substack{K \in \text{ÉtExt}(G,F): \\ \text{lastjump}(K)=N}} \frac{1}{|\text{Aut}(K)|} = C(N \bmod M) \cdot q^{AN} N^{B-1} + o(q^{AN} N^{B-1}).$$

Moreover, if $p \geq 5$, then $B = 1$ (cf. Proposition 6.8).

Remark 1.4. In principle, asymptotics for the number of field extensions (excluding non-simple étale algebras) could be obtained by inclusion–exclusion over subgroups of G . Note also the following criterion for surjectivity, which follows from [MKS04, Lemma 5.9]: when G is a nilpotent group, a homomorphism $\Gamma_F \rightarrow G$ is surjective if and only if the induced homomorphism $\Gamma_F^{\text{ab}} \rightarrow G^{\text{ab}}$ is surjective. (This fact was also used in [KP23].)

1.4. Strategy and outline of the paper. We now summarize the content of each section. This outline also serves as an explanation of our general strategy.

A key tool in our proofs is Abrashkin’s *nilpotent Artin–Schreier theory* from [Abr98] (simpler forms of this theory for p -groups of exponent p and for p -groups of nilpotency class ≤ 2 , were respectively introduced in [Abr95a] and in [Abr95b]). Section 2 is essentially a reformulation of this theory. We review a general principle for the parametrization of extensions (Proposition 2.4), and we explain how to apply this principle using Witt vectors and Lie algebras (Corollary 2.13, Theorem 2.19).

In Section 3, we assume that the base field is a local function field of characteristic p . We refine the parametrization by describing an approximate fundamental domain, making the parametrization finite-to-one (Theorem 3.6), and we use Abrashkin’s description of the ramification filtration from [Abr98] to characterize extensions with a given last jump (Definition 3.9, Theorem 3.20).

In Section 4, we analyze the equations obtained in the previous section in order to obtain bounds on the number of their solutions, and hence on the number of local extensions with bounded last jump (Theorem 4.2). Locally, the bounds we obtain are rough: we have precise estimates of the number of extensions for small values of the last jump, but only upper bounds for large values.

In Section 5, we assume that the base field is a rational global function field of characteristic p . We prove an exact local–global principle for the last jump (Theorem 5.2, which is Theorem 1.1),

as well as a general analytic lemma allowing one to deduce global asymptotics from local estimates (Lemma 5.4). These two tools, combined with the results of the previous section, let us prove our main counting theorem (Theorem 5.6, which is Theorem 1.2) in Subsection 5.3.

Handling the case where the p -torsion subgroup $G[p]$ is non-abelian of size $\geq p^p$ requires a more careful analysis of the equations. In Section 6, we consider the infinite family of (non-abelian) Heisenberg groups $H_k(\mathbb{F}_p)$ (of exponent p), for which the hypotheses of Theorem 1.2 need not be satisfied, and we obtain global asymptotics for the number of $H_k(\mathbb{F}_p)$ -extensions of $\mathbb{F}_q(T)$ (see Theorem 6.7, which is Theorem 1.3). This example illustrates what is needed to solve the problem for more general p -groups of exponent p : a key step is to estimate the number of elements commuting with their Frobenius in a certain Lie algebra.

1.5. Possible improvements. Refinements could come from better understanding the geometry of the varieties defined by the equations of Definition 3.9 (cf. Remark 4.10) in order to improve our bounds on their number of \mathbb{F}_q -points (for example, using the Grothendieck–Lefschetz trace formula as in [ETW23] and [SV22]). Another interesting question, which could be related to the geometric point of view on the problem, is whether the generating functions associated to our counting problems are rational functions, as such a phenomenon was observed in [Gun26] for abelian p -extensions.

Although we consider only groups of nilpotency class ≤ 2 , Abrashkin’s theory applies in principle to all p -groups of nilpotency class $< p$. However, practical difficulties arise when using it for counting purposes due to the complexity of the description of the ramification filtration. The equivalent formulations given by Abrashkin in [Abr23] (in terms of the canonical connection on φ -modules from [Fon90, 2.2.4]) and in [Abr26] might be better suited for counting. Moreover, we have no reason to think that our exact local–global principle (Theorem 5.2) holds for groups of higher nilpotency class.

It should be noted that parametrizing extensions is not the major difficulty (see Subsection 2.2). For instance, $\mathrm{GL}_n(\mathbb{F}_p)$ -extensions are parametrized by étale φ -modules of dimension n , and over local function fields the explicit (“group-theoretic”) description of the absolute Galois group given in [Koc67] (without its ramification filtration!) offers in theory easy access to all extensions. The main challenge lies in obtaining a sufficiently good description of the ramification filtration of these extensions (e.g., an expression for the discriminant/last jump/... in terms of the parametrization) to make counting possible. In that regard, the results of [Ima25] are interesting, as they give an example where the ramification filtration is described even when the nilpotency class equals p , extending slightly beyond the scope of Abrashkin’s theory. For groups of nilpotency class 2, the case $p = 2$ is also mentioned in [Abr10], whereas we do not say anything about the case $p = 2$ in this article.

It is natural to ask whether other invariants satisfy the exact local–global principle from Theorem 1.1. For the discriminant, we doubt this — at least, our method of proof does not apply. Nevertheless, it might be feasible to prove an approximate local–global principle sufficient for a statement analogous to Theorem 1.2.

In our main results, the base field is always a rational (global) function field $\mathbb{F}_q(T)$, with trivial class group. For non-rational base fields $F \neq \mathbb{F}_q(T)$, the exact local–global principle can famously fail even when only considering unramified abelian extensions. This failure can be quantified for abelian p -groups G using Selmer groups (see [Lag12, Lag15, Pot26]) and it seems plausible that the same methods can be used to generalize our main counting results (Theorems 1.2 and 1.3) to non-rational base fields.

1.6. Terminology and notation. If X is a set, we denote its cardinality by $|X|$. When x is an element of a set X on which a group acts, we denote by $[x]$ the orbit of x , usually without specifying which group is acting when the context makes it clear.

Throughout the article, σ always denotes the absolute Frobenius endomorphism $x \mapsto x^p$ of a commutative ring R of characteristic p , and we also call σ the endomorphism induced by σ for every object constructed functorially from R .

In this article, Galois cohomology sets $H^i(\Gamma_F, G)$ are defined using *continuous* group cohomology (absolute Galois groups are equipped with the profinite topology, and G is a topological group on which Γ_F acts continuously). These cohomology sets are themselves groups if G is abelian. If G is non-abelian, they are defined only if $i \in \{0, 1\}$, and are pointed sets with no natural group structure (cf. [Ser62, Chap. VII, Annexe]).

Many notations are introduced in the text. These notations, together with references to their definition and with a short description, are listed in an appendix (p. 40).

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2. PRELIMINARIES

In this section, we review known results concerning the following topics:

- the parametrization of extensions in characteristic p (Subsections 2.1 and 2.2);
- perfect closures of rings in characteristic p (Subsection 2.3);
- Witt vectors and their Galois cohomology (Subsection 2.4);
- the Lazard correspondence, which relates each finite p -group G of nilpotency class $< p$ to a finite Lie \mathbb{Z}_p -algebra (Subsection 2.5);
- the approach developed by Abrashkin under the name *nilpotent Artin-Schreier theory*, which parametrizes G -extensions in characteristic p (Subsection 2.6).

Our explanations loosely follow those given by Abrashkin in [Abr98, §1], but with some differences (cf. Remark 2.20). The theory presented in Subsection 2.6 takes a simpler form when G is a p -group of exponent p , as perfect closures and Witt vectors are not required. See Subsection 6.1 for a brief overview of the simplified theory.

2.1. Extensions and cohomology classes. Let G be a finite group and F be a field. By a G -*extension* of F , we mean an étale F -algebra K together with an action of G such that there is a G -equivariant F^{sep} -algebra isomorphism between $K \otimes_F F^{\text{sep}}$ and the ring of maps $f: G \rightarrow F^{\text{sep}}$ on which G acts via $(g.f)(h) = f(hg)$. An *isomorphism* between two G -extensions of F is a G -equivariant isomorphism between the corresponding étale F -algebras. We denote the set of isomorphism classes of G -extensions of F by $\text{ÉtExt}(G, F)$, often confusing an isomorphism class in $\text{ÉtExt}(G, F)$ with one of its representatives K , and denoting by $\text{Aut}(K)$ the group of its G -equivariant F -algebra automorphisms. We recall the well-known relationship between $\text{ÉtExt}(G, F)$ and the set $H^1(\Gamma_F, G)$ of G -conjugacy classes of continuous group homomorphisms $\gamma \in \text{Hom}(\Gamma_F, G)$, seeing G as a discrete topological group equipped with the trivial Γ_F -action:

Lemma 2.1. *There is a bijection*

$$\text{ÉtExt}(G, F) \xrightarrow{\sim} H^1(\Gamma_F, G)$$

such that, if $K \in \text{ÉtExt}(G, F)$ corresponds to $[\gamma] \in H^1(\Gamma_F, G)$, then:

- (a) For the action of G on $\text{Hom}(\Gamma_F, G)$ by conjugation, we have $\text{Aut}(K) \simeq \text{Stab}_G(\gamma)$.
- (b) K is a field if and only if $\gamma: \Gamma_F \rightarrow G$ is surjective.
- (c) K is the trivial G -extension of F (the ring of maps $G \rightarrow F$) if and only if $\gamma = 1$.

Definition 2.2 (Twisting). Let N be a subgroup of the center of G . The *twist* of $\gamma \in \text{Hom}(\Gamma_F, G)$ by $\delta \in \text{Hom}(\Gamma_F, N)$ is the point-wise product $\gamma \cdot \delta \in \text{Hom}(\Gamma_F, G)$.

Remark 2.3. Denoting the projection $G \twoheadrightarrow G/N$ by π , the twisting operation lets us define a bijection for any given $\gamma \in \text{Hom}(\Gamma_F, G)$:

$$\begin{array}{ccc} \text{Hom}(\Gamma_F, N) & \xrightarrow{\sim} & \{\gamma' \in \text{Hom}(\Gamma_F, G) \mid \pi \circ \gamma' = \pi \circ \gamma\}, \\ \delta & \longmapsto & \gamma \cdot \delta. \end{array}$$

Said differently, twisting defines a free action of the abelian group $\text{Hom}(\Gamma_F, N)$ on the set $\text{Hom}(\Gamma_F, G)$, and each orbit $[\gamma]$ is determined by the (well-defined) element $\pi \circ \gamma \in \text{Hom}(\Gamma_F, G/N)$.

2.2. Parametrization of extensions. We fix a field F . In this subsection, we describe a general principle for parametrizing elements of $H^1(\Gamma_F, G)$.

Let $G_{F^{\text{sep}}}$ be a topological group equipped with a continuous action of Γ_F and with a Γ_F -equivariant group homomorphism $\sigma: G_{F^{\text{sep}}} \rightarrow G_{F^{\text{sep}}}$. We denote by G the subgroup of $G_{F^{\text{sep}}}$ consisting of fixed points of σ , and by G_F the closed subgroup of $G_{F^{\text{sep}}}$ consisting of Γ_F -invariant elements. Note that, as the actions of σ and Γ_F commute, we have $\sigma(G_F) \subseteq G_F$. We define a left action of $G_{F^{\text{sep}}}$ on itself via the formula

$$g.m := \sigma(g)mg^{-1}.$$

This action restricts to an action of G_F on itself, whose set of orbits we denote by $G_F //_{G_F}$. The *multiplicative Artin–Schreier map* is the map $\wp: G_{F^{\text{sep}}} \rightarrow G_{F^{\text{sep}}}$ defined by

$$\wp(g) := \sigma(g)g^{-1} = g.1.$$

Note that $\wp(g) = 1$ if and only if $g \in G$. If $G_{F^{\text{sep}}}$ is abelian, then $g.m = \wp(g)m$ for all $g, m \in G_{F^{\text{sep}}}$, so that the set of orbits $G_F //_{G_F}$ is the quotient group $G_F / \wp(G_F)$.

Proposition 2.4. *Assume that the following properties hold:*

- (i) $G \subseteq G_F$;
- (ii) $G_F \subseteq \wp(G_{F^{\text{sep}}})$;
- (iii) *The map of pointed sets $H^1(\Gamma_F, G) \rightarrow H^1(\Gamma_F, G_{F^{\text{sep}}})$ is trivial.*

Then, there is a bijection

$$\begin{array}{ccc} \text{orb}: & H^1(\Gamma_F, G) & \xrightarrow{\sim} G_F //_{G_F} \\ & [\tau \mapsto g^{-1}\tau(g)] & \longmapsto [\wp(g)] \quad \text{for any } g \in \wp^{-1}(G_F). \end{array}$$

Moreover, if $\text{orb}([\gamma]) = [m]$, then $\text{Stab}_G(\gamma) = \text{Stab}_{G_F}(m)$.

Proof. As in [BG14, Proposition 1], there is a bijection between the kernel of the map of pointed sets $H^1(\Gamma_F, \text{Stab}_{G_{F^{\text{sep}}}}(1)) \rightarrow H^1(\Gamma_F, G_{F^{\text{sep}}})$ and the set $(G_F \cap \wp(G_{F^{\text{sep}}})) //_{G_F}$ of G_F -orbits of elements of $G_F \cap \wp(G_{F^{\text{sep}}})$. For the action of $G_{F^{\text{sep}}}$ on itself, the stabilizer of 1 is G by definition, (i) implies that Γ_F does act trivially on G , (ii) implies that the orbit $\wp(G_{F^{\text{sep}}})$ of 1 contains G_F (so that $(G_F \cap \wp(G_{F^{\text{sep}}})) //_{G_F} = G_F //_{G_F}$), and (iii) implies that $\ker(H^1(\Gamma_F, G) \rightarrow H^1(\Gamma_F, G_{F^{\text{sep}}})) = H^1(\Gamma_F, G)$. We have the desired bijection, and its definition matches the one given here. The equality between stabilizers is easily verified. \square

Assume that F has characteristic p . We illustrate the principle with fundamental examples:

Example 2.5. Let \mathcal{G} be an algebraic group over \mathbb{F}_p and $G = \mathcal{G}(\mathbb{F}_p)$. Then, the group $\mathcal{G}(F^{\text{sep}})$, equipped with its natural Γ_F -action and absolute Frobenius endomorphism σ , is a good candidate to apply Proposition 2.4, as it always satisfies condition (i), and moreover $G_F = \mathcal{G}(F)$. However, conditions (ii) and (iii) still need to be verified.

Example 2.6 (Artin–Schreier theory). Consider the group $G_{F^{\text{sep}}} = F^{\text{sep}}$ (the case $\mathcal{G} = \mathbb{G}_a$ of Example 2.5). The subgroup of Γ_F -invariant elements is $G_F = F$. The subgroup of elements fixed by the Frobenius homomorphism $\sigma(x) = x^p$ is $G = \mathbb{F}_p$. The map $\wp: F^{\text{sep}} \rightarrow F^{\text{sep}}$, $x \mapsto x^p - x$ is surjective, and we have $H^1(\Gamma_F, F^{\text{sep}}) = 0$ by [Ser62, Chap. X, § 1, Prop. 1]. Hence, Proposition 2.4 yields the well-known bijection $H^1(\Gamma_F, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} F/\wp(F)$.

Example 2.7. Let $G = \text{GL}_n(\mathbb{F}_p)$ and $G_{F^{\text{sep}}} = \text{GL}_n(F^{\text{sep}})$ (the case $\mathcal{G} = \text{GL}_n$ of Example 2.5). The subgroup of Γ_F -invariant elements is $\text{GL}_n(F)$. In this case, $H^1(\Gamma_F, G_{F^{\text{sep}}})$ vanishes by a generalization of Hilbert’s Theorem 90 (cf. [Ser62, Chap. X, §1, Prop. 3]), and the map \wp is surjective on F^{sep} -points as it comes from an étale morphism $(\text{GL}_n)_{\mathbb{F}_p} \rightarrow (\text{GL}_n)_{\mathbb{F}_p}$. We retrieve the theory of *étale φ -modules* of dimension n (cf. [FO22, Subsection 3.2], and notably their Remark 3.24). In particular, the case $n = 1$ gives a special case of Kummer theory for the parametrization of $\mathbb{Z}/(p-1)\mathbb{Z}$ -extensions (the case $\mathcal{G} = \mathbb{G}_m$ of Example 2.5).

2.3. Perfect closure. Let R be an integral domain of characteristic p , so that its Frobenius endomorphism $\sigma: R \rightarrow R, x \mapsto x^p$ is injective. The *perfect closure* R^{perf} of R is the direct limit

$$R^{\text{perf}} := \varinjlim (R \xrightarrow{\sigma} R \xrightarrow{\sigma} R \rightarrow \dots).$$

In other words, any element of R^{perf} is a formal p^n -th root of an element of R for some $n \geq 0$. Since $\sigma: R \rightarrow R$ is injective, the canonical map $R \rightarrow R^{\text{perf}}$ is an injection, and we regard R as a subring of R^{perf} via this map. The absolute Frobenius σ of R^{perf} extends that of R and is an automorphism, so R^{perf} is a perfect ring containing R . That construction is functorial in R .

Let now F be a field of characteristic p , and let F^{sep} be a separable closure of F . The field $F^{\text{alg}} := (F^{\text{sep}})^{\text{perf}}$ is an algebraic closure of F , and it is also a separable closure of F^{perf} . When we refer to the absolute Galois group $\Gamma_{F^{\text{perf}}}$, we mean $\text{Gal}(F^{\text{alg}}|F^{\text{perf}})$. A key property of the perfect closure is that its separable extensions correspond bijectively to those of F :

Lemma 2.8. *The restriction map $\Gamma_{F^{\text{perf}}} \rightarrow \Gamma_F$ is an isomorphism of topological groups.*

Proof. The preimage of an automorphism τ of $F^{\text{sep}}|F$ is the automorphism of $F^{\text{alg}}|F^{\text{perf}}$ sending $\sqrt[p^n]{x}$ to $\sqrt[p^n]{\tau(x)}$ for all $x \in F$ and all $n \geq 0$. We leave it to the reader to check that this inverse map is continuous. \square

We let Γ_F act on $F^{\text{alg}} = (F^{\text{perf}})^{\text{sep}}$ via the isomorphism $\Gamma_F \simeq \Gamma_{F^{\text{perf}}}$.

For any finite group G , composition with the restriction map $\Gamma_F \rightarrow \Gamma_{F^{\text{perf}}}$ induces a bijection between the pointed sets $H^1(\Gamma_{F^{\text{perf}}}, G)$ and $H^1(\Gamma_F, G)$, which by Lemma 2.1 means that there is a bijection $\text{ÉtExt}(G, F^{\text{perf}}) \simeq \text{ÉtExt}(G, F)$.

We denote by \wp the \mathbb{F}_p -linear endomorphism $x \mapsto \sigma(x) - x$. For $G = \mathbb{Z}/p\mathbb{Z}$, the bijection $\text{ÉtExt}(\mathbb{Z}/p\mathbb{Z}, F^{\text{perf}}) \simeq \text{ÉtExt}(\mathbb{Z}/p\mathbb{Z}, F)$ turns into a bijection $F^{\text{perf}}/\wp(F^{\text{perf}}) \simeq F/\wp(F)$ using Artin–Schreier theory. This can also be observed directly:

Lemma 2.9. *The map $F/\wp(F) \rightarrow F^{\text{perf}}/\wp(F^{\text{perf}})$ induced by the inclusion $F \subseteq F^{\text{perf}}$ is an isomorphism.*

Proof. The injectivity amounts to the inclusion $F \cap \wp(F^{\text{perf}}) \subseteq \wp(F)$. Let $x \in F$. The equation $y^p - y = x$ is a separable equation in the variable y with coefficients in F . Since the extension $F^{\text{perf}}|F$ is purely inseparable, any solution y in F^{perf} has to lie in F . Therefore, if $x \in \wp(F^{\text{perf}})$, then $x \in \wp(F)$.

We now check surjectivity. Let $x \in F^{\text{perf}}$. By definition of F^{perf} , there is some $n \geq 0$ such that $\sigma^n(x) \in F$. By definition of \wp , we have $z \equiv \sigma(z) \pmod{\wp(F^{\text{perf}})}$ for all $z \in F^{\text{perf}}$, and in particular $x \equiv \sigma(x) \equiv \dots \equiv \sigma^n(x)$. \square

2.4. Witt vectors. To deal with p -groups of exponent larger than p , we use p -typical Witt vectors. We have a functor

$$W: \{\text{rings of characteristic } p\} \longrightarrow \{\mathbb{Z}_p\text{-algebras}\}$$

mapping a ring R to the corresponding ring of Witt vectors $W(R)$, whose elements can be represented as vectors of infinite length with coordinates in R . Let R be an integral domain of characteristic p . The operation consisting of adding a leading zero to a Witt vector, shifting all other coordinates one place to the right, defines the (\mathbb{Z}_p -linear) Verschiebung operator $\text{Ver}: W(R) \rightarrow W(R)$. Moreover, the absolute Frobenius endomorphism $\sigma: x \mapsto x^p$ of R induces a coordinatewise ring endomorphism σ of $W(R)$, fixing exactly the elements of $W(\mathbb{F}_p) = \mathbb{Z}_p$. We denote by \wp the \mathbb{Z}_p -linear map $W(R) \rightarrow W(R), x \mapsto \sigma(x) - x$, whose kernel is \mathbb{Z}_p . The endomorphism of R given by multiplication by p coincides with $\text{Ver} \circ \sigma = \sigma \circ \text{Ver}$. The ring of Witt vectors of length n over R is the $\mathbb{Z}/p^n\mathbb{Z}$ -algebra

$$W_n(R) := W(R)/\text{Ver}^n(W(R)),$$

which coincides with $W(R)/p^n W(R)$ when R is perfect. We have $W(R) = \varprojlim_n W_n(R)$.

We now fix a field F of characteristic p . The action of Γ_F on $F^{\text{alg}} = (F^{\text{sep}})^{\text{perf}}$ induces a coordinatewise action on $W(F^{\text{alg}})$, fixing exactly the elements of $W(F^{\text{perf}})$.

Remark 2.10 (Artin–Schreier–Witt theory). The additive group $G_{F^{\text{sep}}} := W_n(F^{\text{sep}})$ satisfies the hypotheses of Proposition 2.4 with $G_F = W_n(F)$ and $G = W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}$. Hence, we have a bijection $H^1(\Gamma_F, \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\sim} W_n(F)/\wp(W_n(F))$. (See for example [Bos18, Lemmas 9 and 10, and Proposition 11 in Section 4.10].)

Remark 2.11. The ring $W(F^{\text{perf}})$ is a torsion-free \mathbb{Z}_p -module, hence flat, so any short exact sequence of \mathbb{Z}_p -modules $0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{n} \rightarrow 0$ induces an exact sequence of $W(F^{\text{perf}})$ -modules

$$0 \rightarrow \mathfrak{n} \otimes_{\mathbb{Z}_p} W(F^{\text{perf}}) \rightarrow \mathfrak{g} \otimes_{\mathbb{Z}_p} W(F^{\text{perf}}) \rightarrow (\mathfrak{g}/\mathfrak{n}) \otimes_{\mathbb{Z}_p} W(F^{\text{perf}}) \rightarrow 0.$$

Lemma 2.12. *The following properties hold for any finite \mathbb{Z}_p -module \mathfrak{g} :*

- (i) *The Artin–Schreier map $\wp : \mathfrak{g} \otimes W(F^{\text{alg}}) \rightarrow \mathfrak{g} \otimes W(F^{\text{alg}})$, $g \mapsto \sigma(g) - g$, is surjective.*
- (ii) *The natural map $\mathfrak{g} \otimes W(F^{\text{perf}}) \rightarrow \mathfrak{g} \otimes W(F^{\text{alg}})$ is injective.*
- (iii) *$(\mathfrak{g} \otimes W(F^{\text{alg}}))^{\Gamma_F} = \mathfrak{g} \otimes W(F^{\text{perf}})$.*
- (iv) *$(\mathfrak{g} \otimes W(F^{\text{alg}}))^{\sigma} = \mathfrak{g}$.*
- (v) *$H^1(\Gamma_F, \mathfrak{g} \otimes W(F^{\text{alg}})) = 0$.*

Proof. We can assume without loss of generality that $\mathfrak{g} = \mathbb{Z}/p^n\mathbb{Z}$ with $n \geq 1$, as every finite \mathbb{Z}_p -module is a direct sum of such factors. We have $\mathfrak{g} \otimes W(F^{\text{perf}}) = W_n(F^{\text{perf}})$ and $\mathfrak{g} \otimes W(F^{\text{alg}}) = W_n(F^{\text{alg}})$. Then:

- (i) Apply [Bos18, Lemma 9 in Section 4.10] to $K = F^{\text{perf}}$ (recall that $F^{\text{alg}} = (F^{\text{perf}})^{\text{sep}}$).
- (ii) Clear.
- (iii) Clear (the action of $\Gamma_F \simeq \Gamma_{F^{\text{perf}}}$ on $W_n(F^{\text{alg}})$ is coordinatewise).
- (iv) Clear (the action of σ on $W_n(F^{\text{alg}})$ is coordinatewise, and $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}$).
- (v) This follows from [Bos18, Proposition 11 in Section 4.10], applied to $K = F^{\text{perf}}$. \square

Lemma 2.12 lets us apply Proposition 2.4 to obtain the following parametrization of G -extensions of F , where G is any finite abelian p -group (corresponding to $(\mathfrak{g}, +)$ for a finite \mathbb{Z}_p -module \mathfrak{g}):

Corollary 2.13. *Let \mathfrak{g} be a finite \mathbb{Z}_p -module, and let $\wp : \mathfrak{g} \otimes W(F^{\text{perf}}) \rightarrow \mathfrak{g} \otimes W(F^{\text{perf}})$ be the (additive) Artin–Schreier map $g \mapsto \sigma(g) - g$. We have a bijection*

$$H^1(\Gamma_F, (\mathfrak{g}, +)) \xrightarrow{\sim} \mathfrak{g} \otimes W(F^{\text{perf}})/\wp(\mathfrak{g} \otimes W(F^{\text{perf}})).$$

The goal of the following subsections is to obtain a non-abelian version of Corollary 2.13 — this will be Theorem 2.19.

2.5. Lie \mathbb{Z}_p -algebras.

2.5.1. Definitions. A *Lie \mathbb{Z}_p -algebra* is a \mathbb{Z}_p -module \mathfrak{g} equipped with a (\mathbb{Z}_p -bilinear, alternating) Lie bracket $[-, -] : \mathfrak{g}^2 \rightarrow \mathfrak{g}$ satisfying the Jacobi identity $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$. Let \mathfrak{g} be a Lie \mathbb{Z}_p -algebra. We say that \mathfrak{g} is *abelian* if its Lie bracket is identically zero. An *ideal* of \mathfrak{g} is a submodule $\mathfrak{n} \subseteq \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{n}] \subseteq \mathfrak{n}$. We can form the quotient of \mathfrak{g} by an ideal \mathfrak{n} to obtain a Lie algebra $\mathfrak{g}/\mathfrak{n}$. The *center* of \mathfrak{g} is the ideal $Z(\mathfrak{g})$ formed of elements x such that $[\mathfrak{g}, x] = 0$. For elements $x_1, \dots, x_n \in \mathfrak{g}$, we use the notation

$$[x_1, \dots, x_n] := \underbrace{[[\dots [x_1, x_2], x_3], \dots, x_n]}_{n-1}.$$

We say that \mathfrak{g} is *nilpotent* if there is an integer n such that $[x_1, \dots, x_{n+1}]$ vanishes for all $x_1, \dots, x_{n+1} \in \mathfrak{g}$. The smallest such n is then the *nilpotency class* of \mathfrak{g} . For instance, the zero Lie algebra is the unique Lie algebra of class 0, and the Lie algebras of class 1 are exactly the non-zero abelian Lie algebras. The center of a non-zero nilpotent Lie algebra \mathfrak{g} is a non-zero abelian Lie algebra. In particular, if $\mathfrak{g} \neq 0$ is finite and nilpotent, then there is a Lie subalgebra $\mathfrak{n} \subseteq Z(\mathfrak{g})$ with $\mathfrak{n} \simeq \mathbb{Z}/p\mathbb{Z}$.

Lie algebras of nilpotency class ≤ 2 are those for which the Lie bracket is valued in the center, i.e., such that $[\mathfrak{g}, \mathfrak{g}] \subseteq Z(\mathfrak{g})$. If \mathfrak{g} is a \mathbb{Z}_p -module and \mathfrak{z} is a given submodule of \mathfrak{g} , then equipping \mathfrak{g} with a Lie bracket such that \mathfrak{g} has nilpotency class ≤ 2 and center \mathfrak{z} amounts to

giving a non-degenerate alternating \mathbb{Z}_p -bilinear map $(\mathfrak{g}/\mathfrak{z}) \oplus (\mathfrak{g}/\mathfrak{z}) \rightarrow \mathfrak{z}$ (the Jacobi identity is automatically satisfied).

2.5.2. *The Lazard correspondence* (cf. [Laz54]). Let \mathfrak{g} be a Lie \mathbb{Z}_p -algebra of nilpotency class $< p$. We define a group law \circ on \mathfrak{g} via the *truncated Baker–Campbell–Hausdorff formula*

$$x \circ y := x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, y, y] - \frac{1}{12}[x, y, x] + \dots$$

where the sum is including only the finitely many terms of the Baker–Campbell–Hausdorff formula (see e.g. [Ser06, p. 29]) which do not feature p -fold commutators, thus involving only denominators coprime to p and making the sum well-defined. For instance, for Lie algebras of nilpotency class ≤ 2 (with $p > 2$), the formula simplifies to $x \circ y = x + y + \frac{1}{2}[x, y]$. The operation transforming the Lie algebra \mathfrak{g} into the group (\mathfrak{g}, \circ) is the key construction in the *Lazard correspondence*, which is an equivalence of categories

$$\{\text{finite Lie } \mathbb{Z}_p\text{-algebras of nilpotency class } < p\} \longleftrightarrow \{\text{finite } p\text{-groups of nilpotency class } < p\}.$$

In particular, every finite p -group G of nilpotency class $< p$ is isomorphic to (\mathfrak{g}, \circ) for some finite Lie \mathbb{Z}_p -algebra \mathfrak{g} , which then has the same nilpotency class. This correspondence was introduced by Lazard in [Laz54] (see also [CdGVL12] or [Abr98, Section 1.2]), and it is somewhat analogous to the classical Lie correspondence between Lie algebras and Lie groups.

Via this correspondence, Lie subalgebras of \mathfrak{g} correspond to subgroups of (\mathfrak{g}, \circ) . Similarly, ideals of \mathfrak{g} correspond to normal subgroups of (\mathfrak{g}, \circ) , and the quotients then correspond to each other: a short exact sequence $0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{n} \rightarrow 0$ of Lie algebras (i.e., a short exact sequence of \mathbb{Z}_p -modules in which every arrow is a Lie algebra homomorphism) induces a short exact sequence $1 \rightarrow (\mathfrak{n}, \circ) \rightarrow (\mathfrak{g}, \circ) \rightarrow (\mathfrak{g}/\mathfrak{n}, \circ) \rightarrow 1$ of groups.

As the Lie bracket of an element $x \in \mathfrak{g}$ with itself vanishes, the n -th power of x as an element of the group (\mathfrak{g}, \circ) is $n \cdot x$, for all $n \in \mathbb{Z}$. In particular, the inverse of x with respect to \circ is simply its additive inverse $-x$.

If $[x, y] = 0$, then $x \circ y = y \circ x = x + y$. Conversely, if x and y commute in (\mathfrak{g}, \circ) , then the Campbell identity

$$x \circ y \circ (-x) = \sum_{n=0}^{p-1} \frac{(-1)^n}{n!} [y, \underbrace{x, \dots, x}_n]$$

implies that $[x, y]$ is an n -fold commutator for any $n \geq 2$ (by induction on n) and hence $[x, y] = 0$ as \mathfrak{g} is nilpotent. In particular, the center $Z(\mathfrak{g})$ of the Lie algebra \mathfrak{g} coincides with the center of the group (\mathfrak{g}, \circ) . Note that the Lie algebra \mathfrak{g} is abelian if and only if the group (\mathfrak{g}, \circ) is abelian, in which case the laws $+$ and \circ coincide.

If \mathfrak{g} is a \mathbb{Z}_p -module, the set $\mathfrak{g}[p]$ of its p -torsion elements forms an \mathbb{F}_p -vector space. When \mathfrak{g} is a Lie \mathbb{Z}_p -algebra, the Lie \mathbb{F}_p -algebra $\mathfrak{g}[p]$ is an ideal of \mathfrak{g} : indeed, if $x \in \mathfrak{g}[p]$, then $p \cdot [x, y] = [p \cdot x, y] = 0$ for any $y \in \mathfrak{g}$. Using the Lazard correspondence, this directly implies:

Lemma 2.14. *Let G be a finite p -group of nilpotency class smaller than p . Its p -torsion elements form a normal subgroup of G , which we denote by $G[p]$.*

Remark 2.15. The condition on the nilpotency class is crucial in Lemma 2.14: for example, the 2-torsion elements of the 2-group D_4 (of nilpotency class 2) do not form a subgroup of D_4 .

2.6. Nilpotent Artin–Schreier theory. We fix a finite Lie \mathbb{Z}_p -algebra \mathfrak{g} of nilpotency class smaller than p , and we let $G := (\mathfrak{g}, \circ)$. (By Paragraph 2.5.2, every finite p -group G of nilpotency class $< p$ arises in this way.) The tensor product $\mathfrak{g} \otimes_{\mathbb{Z}_p} W(F^{\text{alg}})$ inherits a Lie \mathbb{Z}_p -algebra structure from \mathfrak{g} (and hence a group law \circ) by $W(F^{\text{alg}})$ -linear extension. In this subsection, we apply Proposition 2.4 to the group $G_{F^{\text{sep}}} := (\mathfrak{g} \otimes W(F^{\text{alg}}), \circ)$ in order to prove Theorem 2.19.

The actions of Γ_F and of σ on $\mathfrak{g} \otimes W(F^{\text{alg}})$ respect the group law \circ , so they are actions on the group $G_{F^{\text{sep}}}$, and they commute with each other. By Lemma 2.12 (points (iii) and (iv)), the subgroup of $G_{F^{\text{sep}}}$ fixed by Γ_F is $G_F := (\mathfrak{g} \otimes W(F^{\text{perf}}), \circ)$ and the subgroup fixed by σ is $G = (\mathfrak{g}, \circ)$. Following Subsection 2.2, we then define a left action of the group $G_{F^{\text{sep}}}$ on the set $\mathfrak{g} \otimes W(F^{\text{alg}})$ by $g.m := \sigma(g) \circ m \circ (-g)$. This action restricts to an action of the

subgroup $G_F = (\mathfrak{g} \otimes W(F^{\text{perf}}), \circ)$ on the set $\mathfrak{g} \otimes W(F^{\text{perf}})$. We write $\mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$ for the set of $(\mathfrak{g} \otimes W(F^{\text{perf}}), \circ)$ -orbits of $\mathfrak{g} \otimes W(F^{\text{perf}})$. The multiplicative Artin–Schreier map $\wp : \mathfrak{g} \otimes W(F^{\text{alg}}) \rightarrow \mathfrak{g} \otimes W(F^{\text{alg}})$ is given by $g \mapsto \sigma(g) \circ (-g)$, so that $\wp(g) = g \cdot 0$.

We first prove the following lemma, which is used in proofs by induction on the size of \mathfrak{g} :

Lemma 2.16. *Let $g, m \in \mathfrak{g} \otimes W(F^{\text{alg}})$ and $h, n \in Z(\mathfrak{g}) \otimes W(F^{\text{alg}})$. Then, $(g + h) \cdot (m + n) = g \cdot m + h \cdot n$.*

Proof.

$$\begin{aligned} (g + h) \cdot (m + n) &= \sigma(g + h) \circ (m + n) \circ (-g - h) \\ &= \sigma(g) \circ \sigma(h) \circ m \circ n \circ (-g) \circ (-h) && \text{as } h, n \in Z(\mathfrak{g}) \otimes W(F^{\text{alg}}) \\ &= \sigma(g) \circ m \circ (-g) \circ \sigma(h) \circ n \circ (-h) && \text{as } \sigma(h), n \in Z(\mathfrak{g}) \otimes W(F^{\text{alg}}) \\ &= (g \cdot m) \circ (h \cdot n) \\ &= g \cdot m + h \cdot n && \text{as } h \cdot n \in Z(\mathfrak{g}) \otimes W(F^{\text{alg}}). \quad \square \end{aligned}$$

In particular, applied to $m = n = 0$, Lemma 2.16 implies:

Corollary 2.17. *Let $g \in \mathfrak{g} \otimes W(F^{\text{alg}})$ and $h \in Z(\mathfrak{g}) \otimes W(F^{\text{alg}})$. Then, $\wp(g + h) = \wp(g) + \wp(h)$.*

We now verify the remaining hypotheses of Proposition 2.4:

Lemma 2.18. *The group $(\mathfrak{g} \otimes W(F^{\text{alg}}), \circ)$ satisfies the following properties:*

- (i) *We have $(\mathfrak{g} \otimes W(F^{\text{alg}}))^\sigma \subseteq (\mathfrak{g} \otimes W(F^{\text{alg}}))^{\Gamma_F}$.*
- (ii) *The multiplicative Artin–Schreier map $\wp : \mathfrak{g} \otimes W(F^{\text{alg}}) \rightarrow \mathfrak{g} \otimes W(F^{\text{alg}})$ is surjective.*
- (iii) *The cohomology set $H^1(\Gamma_F, (\mathfrak{g} \otimes W(F^{\text{alg}}), \circ))$ is a singleton.*

Proof. Point (i) directly follows from Lemma 2.12 (points (iii) and (iv)), so we focus on points (ii) and (iii).

When \mathfrak{g} is an abelian Lie \mathbb{Z}_p -algebra, the group law \circ coincides with $+$, so we have already proved the claims in Lemma 2.12 (points (i) and (v)).

We now prove the result for a general Lie \mathbb{Z}_p -algebra \mathfrak{g} of nilpotency class $< p$, by induction on the size of \mathfrak{g} . The case $\mathfrak{g} = 0$ is clear, so assume that \mathfrak{g} is non-zero and that the result holds for Lie \mathbb{Z}_p -algebras of nilpotency class $< p$ whose size is smaller than $|\mathfrak{g}|$. Pick a subalgebra $\mathfrak{n} \subseteq Z(\mathfrak{g})$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$. We have already shown the claims for the abelian Lie algebra \mathfrak{n} , and the claims hold for the Lie algebra $\mathfrak{g}/\mathfrak{n}$ by the induction hypothesis. To combine these two known cases, we use the exact sequence of Lie \mathbb{Z}_p -algebras

$$0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{n} \rightarrow 0,$$

which induces an exact sequence of Lie \mathbb{Z}_p -algebras (see Remark 2.11)

$$0 \rightarrow \mathfrak{n} \otimes W(F^{\text{alg}}) \rightarrow \mathfrak{g} \otimes W(F^{\text{alg}}) \rightarrow (\mathfrak{g}/\mathfrak{n}) \otimes W(F^{\text{alg}}) \rightarrow 0,$$

and thus an exact sequence of groups (see Paragraph 2.5.2)

$$1 \rightarrow (\mathfrak{n} \otimes W(F^{\text{alg}}), \circ) \rightarrow (\mathfrak{g} \otimes W(F^{\text{alg}}), \circ) \rightarrow ((\mathfrak{g}/\mathfrak{n}) \otimes W(F^{\text{alg}}), \circ) \rightarrow 1.$$

We then prove points (ii) and (iii):

- (ii) Let $x \in \mathfrak{g} \otimes W(F^{\text{alg}})$, and let $\bar{x} \in (\mathfrak{g}/\mathfrak{n}) \otimes W(F^{\text{alg}})$ be its projection. By the induction hypothesis, there is some $\bar{y} \in (\mathfrak{g}/\mathfrak{n}) \otimes W(F^{\text{alg}})$ such that $\wp(\bar{y}) = \bar{x}$. Choose an arbitrary lift $y \in \mathfrak{g} \otimes W(F^{\text{alg}})$ of \bar{y} . Then, $x - \wp(y)$ belongs to $\mathfrak{n} \otimes W(F^{\text{alg}})$. By the case $\mathfrak{g} = \mathbb{Z}/p\mathbb{Z}$, there is a $z \in \mathfrak{n} \otimes W(F^{\text{alg}})$ such that $\wp(z) = x - \wp(y)$. As z is central, Corollary 2.17 implies that $\wp(y + z) = \wp(y) + \wp(z) = x$. Thus, \wp is surjective.

- (iii) In non-abelian Galois cohomology, we have the exact sequence of pointed sets

$$H^1(\Gamma_F, (\mathfrak{n} \otimes W(F^{\text{alg}}), \circ)) \longrightarrow H^1(\Gamma_F, (\mathfrak{g} \otimes W(F^{\text{alg}}), \circ)) \longrightarrow H^1(\Gamma_F, ((\mathfrak{g}/\mathfrak{n}) \otimes W(F^{\text{alg}}), \circ))$$

in which we already know that the cohomology sets associated to \mathfrak{n} and to $\mathfrak{g}/\mathfrak{n}$ are trivial. Hence, so is the cohomology set $H^1(\Gamma_F, (\mathfrak{g} \otimes W(F^{\text{alg}}), \circ))$ associated to \mathfrak{g} . \square

Lemmas 2.12 and 2.18 let us apply Proposition 2.4 to parametrize G -extensions of F where $G = (\mathfrak{g}, \circ)$, generalizing Corollary 2.13 to (some) non-abelian p -extensions:

Theorem 2.19. *There is a bijection*

$$\text{orb}: \quad \begin{array}{ccc} H^1(\Gamma_F, G) & \xrightarrow{\sim} & \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \\ [\tau \mapsto (-g) \circ \tau(g)] & \mapsto & [\wp(g)] \end{array} \quad \text{for any } g \in \wp^{-1}(\mathfrak{g} \otimes W(F^{\text{perf}})).$$

Moreover, if $\text{orb}([\gamma]) = [m]$, then $\text{Stab}_G(\gamma) \simeq \text{Stab}_{(\mathfrak{g} \otimes W(F^{\text{perf}}), \circ)}(m)$.

Remark 2.20. In [Abr98], Abrashkin instead constructed a closely related bijection

$$H^1(\Gamma_F, G) \xrightarrow{\sim} \mathfrak{g} \otimes O(F) //_{O(F)},$$

where $O(F) \subseteq W(F)$ is the Cohen ring of F , i.e., the p -adically complete flat local \mathbb{Z}_p -algebra with maximal ideal p and residue field F (unique up to isomorphism). We refer to [FO22, Subsection 1.2.4] or [BM90, Subsection 1.1] for more details about Cohen rings. The main advantage of Cohen rings over Witt vectors is the fact that $\mathbb{Z}/p\mathbb{Z} \otimes O(F) = O(F)/pO(F) = F$, whereas $\mathbb{Z}/p\mathbb{Z} \otimes W(F^{\text{perf}}) = F^{\text{perf}}$ is slightly larger. (However, this disadvantage disappears modulo the image of the Artin–Schreier map, see Lemma 2.9.) On the other hand, the main advantage of Witt vectors over Cohen rings is functoriality: any field homomorphism $F_1 \hookrightarrow F_2$ induces a canonical ring homomorphism $W(F_1^{\text{perf}}) \hookrightarrow W(F_2^{\text{perf}})$, but not necessarily a canonical ring homomorphism $O(F_1) \rightarrow O(F_2)$. Functoriality is a convenient property for us to apply to the actions of Γ_F and σ on F^{sep} and to embeddings of global fields into their completions.

Proposition 2.21 (Naturality). *The bijection of Theorem 2.19 is natural in both \mathfrak{g} and F :*

- (a) *Let $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be a morphism of finite Lie \mathbb{Z}_p -algebras of nilpotency class smaller than p . We obtain a commutative diagram*

$$\begin{array}{ccc} H^1(\Gamma_F, (\mathfrak{g}_1, \circ)) & \xrightarrow[\sim]{\text{orb}} & \mathfrak{g}_1 \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \\ \downarrow & & \downarrow \\ H^1(\Gamma_F, (\mathfrak{g}_2, \circ)) & \xrightarrow[\sim]{\text{orb}} & \mathfrak{g}_2 \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \end{array}$$

where the vertical maps are the natural maps induced by f .

- (b) *Let $f : F_1 \hookrightarrow F_2$ be a field homomorphism between fields of characteristic p . We obtain a commutative diagram*

$$\begin{array}{ccc} H^1(\Gamma_{F_1}, (\mathfrak{g}, \circ)) & \xrightarrow[\sim]{\text{orb}} & \mathfrak{g} \otimes W(F_1^{\text{perf}}) //_{W(F_1^{\text{perf}})} \\ \downarrow & & \downarrow \\ H^1(\Gamma_{F_2}, (\mathfrak{g}, \circ)) & \xrightarrow[\sim]{\text{orb}} & \mathfrak{g} \otimes W(F_2^{\text{perf}}) //_{W(F_2^{\text{perf}})} \end{array}$$

where the vertical maps are the natural maps induced by f .

Remark 2.22. Let $[m] \in \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$ correspond to a G -extension $K|F$ via the bijections of Theorem 2.19 and Lemma 2.1. Then, Proposition 2.21 implies the following:

- (a) For any ideal \mathfrak{n} of \mathfrak{g} , the orbit $[m \bmod \mathfrak{n}] \in (\mathfrak{g}/\mathfrak{n}) \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$ corresponds to the $(\mathfrak{g}/\mathfrak{n}, \circ)$ -subextension of K fixed by (\mathfrak{n}, \circ) .
- (b) For any valuation v of F , denoting by F_v the completion of F with respect to v , the orbit $[m] \in \mathfrak{g} \otimes W(F_v^{\text{perf}}) //_{W(F_v^{\text{perf}})}$ corresponds to the G -extension $K \otimes_F F_v$ of F_v .

By Theorem 2.19 and Lemma 2.1, elements of $\mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$ correspond bijectively to G -extensions of F . Thus, we extend the definition of the last jump:

Definition 2.23. Assume that F is either a local or a global function field of characteristic p . If $K|F$ is the G -extension associated to an orbit $[m] \in \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$, we define both $\text{lastjump}(m)$ and $\text{lastjump}([m])$ to be $\text{lastjump}(K|F)$ (cf. Subsection 1.2).

Finally, we describe the effect of the “twisting operation” of Definition 2.2 in terms of the parametrization:

Lemma 2.24 (Twisting). *Let \mathfrak{n} be a Lie subalgebra of $Z(\mathfrak{g})$. For any $\gamma \in \text{Hom}(\Gamma_F, (\mathfrak{g}, \circ))$ and $\delta \in \text{Hom}(\Gamma_F, (\mathfrak{n}, \circ))$, let $m \in \mathfrak{g} \otimes W(F^{\text{perf}})$ and $n \in \mathfrak{n} \otimes W(F^{\text{perf}})$ be such that*

$$\text{orb}([\gamma]) = [m] \in \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \quad \text{and} \quad \text{orb}([\delta]) = [n] \in \mathfrak{n} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}.$$

Then, $\text{orb}([\gamma \cdot \delta]) = [m + n] \in \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$.

Proof. Pick $g \in \mathfrak{g} \otimes W(F^{\text{alg}})$ and $h \in \mathfrak{n} \otimes W(F^{\text{alg}})$ such that $m = \wp(g)$ and $n = \wp(h)$, and such that $\gamma(\tau) = (-g) \circ \tau(g)$ and $\delta(\tau) = (-h) \circ \tau(h)$ for all $\tau \in \Gamma_F$. For all $\tau \in \Gamma_F$, we have $(\gamma \cdot \delta)(\tau) = \gamma(\tau) \circ \delta(\tau) = (-g) \circ \tau(g) \circ (-h) \circ \tau(h) = (-g - h) \circ \tau(g + h)$ as $h \in Z(\mathfrak{g}) \otimes W(F^{\text{alg}})$. Moreover, by Corollary 2.17, we have $\wp(g + h) = \wp(g) + \wp(h) = m + n$. Hence, $\text{orb}([\gamma \cdot \delta]) = [m + n]$ by definition. \square

3. LOCAL PARAMETRIZATION

In this section, we fix a finite field κ of characteristic $p > 2$, a local function field \mathfrak{F} with residue field κ , and a uniformizer π of \mathfrak{F} , so that $\mathfrak{F} = \kappa((\pi))$. We denote by $\tilde{\pi}$ the Teichmüller representative $(\pi, 0, 0, \dots) \in W(\mathfrak{F}) \subseteq W(\mathfrak{F}^{\text{perf}})$ of π . We also fix a finite Lie \mathbb{Z}_p -algebra $\mathfrak{g} \neq 0$ of nilpotency class at most 2, and we denote by $G := (\mathfrak{g}, \circ)$ its associated p -group.

Our goal is to study G -extensions of the local field \mathfrak{F} . We refine the parametrization of these extensions by describing “approximate fundamental domains” (Subsections 3.1 and 3.3), and we use Abrashkin’s results from [Abr98] to describe the last jump in terms of this parametrization (Subsections 3.2 and 3.4). This will allow us to count extensions in Section 4.

The main results of this section can be summed up as follows: we define $W(\kappa)$ -modules $\mathcal{D}^0, \mathcal{D} \subseteq W(\mathfrak{F}^{\text{perf}})$ and a surjection $\text{pr}: \mathcal{D}^0 \rightarrow \mathcal{D}$ [cf. Definitions 3.2 and 3.14] such that:

- The action of $(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)$ on $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ restricts to an action of the finite group $(\mathfrak{g} \otimes W(\kappa), \circ)$ on $\mathfrak{g} \otimes \mathcal{D}^0$. We denote by $\mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)}$ the set of orbits for the restricted action. [cf. Remark 3.5]
- The inclusion $\mathcal{D}^0 \subseteq W(\mathfrak{F}^{\text{perf}})$ induces a bijection $\mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)} \xrightarrow{\sim} \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})}$ (the latter is itself in bijection with $\text{ÉtExt}(G, \mathfrak{F})$ by Theorem 2.19). [cf. Theorem 3.6]
- The last jump of an element $D \in \mathfrak{g} \otimes \mathcal{D}^0 \subseteq \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ (in the sense of Definition 2.23) is characterized in terms of its “coordinates” $D_a \in \mathfrak{g} \otimes W(\kappa)$ by the equations of Definition 3.9. [cf. Theorem 3.20]
- There is an action of $(\mathfrak{g} \otimes W(\kappa), \circ)$ on $\mathfrak{g} \otimes \mathcal{D}$ such that the surjection pr induces a surjection $\text{pr}: \mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)} \rightarrow \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}$ between the respective sets of orbits. [cf. Proposition 3.16]
- Each fiber of the surjection $\text{pr}: \mathfrak{g} \otimes \mathcal{D}^0 \rightarrow \mathfrak{g} \otimes \mathcal{D}$ has size $|\mathfrak{g} \otimes W(\kappa)|$ and is formed of elements having the same last jump. [cf. Definition 3.15, Corollary 3.23]

This is summarized by the following diagram:

$$\begin{array}{ccccc} \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) & \longleftarrow & \mathfrak{g} \otimes \mathcal{D}^0 & \xrightarrow{|\mathfrak{g} \otimes W(\kappa)|:1} & \mathfrak{g} \otimes \mathcal{D} \\ \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})} & \xrightarrow{\sim} & \mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)} & \longrightarrow & \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)} \\ \text{lastjump}([m]) & = & \text{lastjump}([D^0]) & = & \text{lastjump}([D]) \end{array}$$

Hence, counting G -extensions $K|\mathfrak{F}$ with $\text{lastjump}(K) < v$ essentially amounts to counting elements of $\mathfrak{g} \otimes \mathcal{D}$ satisfying certain equations (given in Definition 3.9). This fact, which is made more precise in Lemma 4.1, will be used throughout Section 4 to count local extensions.

3.1. Fundamental domain. In Theorem 2.19, we have constructed a bijection between the set $H^1(\Gamma_{\mathfrak{F}}, G)$ and the set $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})}$ of $(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)$ -orbits of $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$. For counting orbits, it is often convenient to work with a fundamental domain consisting of exactly one representative from each orbit. Here, we do a little less: we define a canonical subset $\mathfrak{g} \otimes \mathcal{D}^0$ of $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ which is a fundamental domain up to the action of the finite subgroup $(\mathfrak{g} \otimes W(\kappa), \circ)$ of $(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)$ (see Theorem 3.6). This allows us to count orbits

using the orbit-stabilizer theorem. This “fundamental domain” is closely related to Abrashkin’s notion of “special elements” (cf. [Abr23, Definition 2.1]) and to [Ima25, Lemma 3.2].

Lemma 3.1. *The set $\mathfrak{g} \otimes W(\kappa)$ is finite of size $|\kappa|^n$ if $|\mathfrak{g}| = p^n$.*

Proof. As in the proof of Lemma 2.12, we may assume that $\mathfrak{g} \simeq \mathbb{Z}/p^n\mathbb{Z}$. Then, $\mathfrak{g} \otimes W(\kappa) = W(\kappa)/p^n W(\kappa) = W_n(\kappa)$ since the finite field κ is perfect. The ring $W_n(\kappa)$ of Witt vectors of length n over κ has size $|\kappa|^n$. \square

Definition 3.2. We define the following free $W(\kappa)$ -submodule of $W(\mathfrak{F}) \subseteq W(\mathfrak{F}^{\text{perf}})$:

$$\mathcal{D}^0 := \bigoplus_{a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}} W(\kappa) \tilde{\pi}^{-a}.$$

Note that $\mathcal{D}^0/p\mathcal{D}^0 = \bigoplus_{a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}} \kappa \pi^{-a} \subseteq \mathfrak{F} \subseteq \mathfrak{F}^{\text{perf}}$. The set $\mathfrak{g} \otimes \mathcal{D}^0$ is the sub- $W(\kappa)$ -module of $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ consisting of elements D of the form $\sum_{a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}} D_a \tilde{\pi}^{-a}$, where the coordinates D_a belong to the Lie $W(\kappa)$ -algebra $\mathfrak{g} \otimes W(\kappa)$ and are almost all zero.

Lemma 3.3.

- (i) *If $x \in \mathfrak{F}^{\text{perf}}$ is such that $\wp(x) \in \mathcal{D}^0/p\mathcal{D}^0$, then $x \in \kappa$.*
- (ii) *The map $\mathcal{D}^0/p\mathcal{D}^0 \rightarrow \mathfrak{F}/\wp(\mathfrak{F})$ induced by the inclusion $\mathcal{D}^0/p\mathcal{D}^0 \subseteq \mathfrak{F}$ is a surjection.*

Proof.

- (i) Since $\wp(x) = x^p - x$ lies in \mathfrak{F} , and since $\mathfrak{F}^{\text{perf}}$ is a purely inseparable extension of \mathfrak{F} , we have $x \in \mathfrak{F}$. If the (π -adic) valuation of x is negative, then the valuation of $\wp(x) = x^p - x$ is p times that of x , hence is a non-zero multiple of p , contradicting $\wp(x) \in \mathcal{D}^0/p\mathcal{D}^0$. Thus, $x \in \kappa[[\pi]]$. Assume that $x \notin \kappa$, and write $x = x_0 + x_1$, where $x_0 \in \kappa$ and x_1 has positive valuation. Then $\wp(x_1) = x_1^p - x_1$ has valuation equal to that of x_1 , which is positive, contradicting the fact that $\wp(x_1) = \wp(x) - \wp(x_0)$ belongs to $\mathcal{D}^0/p\mathcal{D}^0$.
- (ii) This is [Pot26, Proposition 5.2 (b)]. \square

Lemma 3.4.

- (i) *Let $D \in \mathfrak{g} \otimes \mathcal{D}^0$ and $g \in \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$. Then:*

$$g.D \in \mathfrak{g} \otimes \mathcal{D}^0 \iff g \in \mathfrak{g} \otimes W(\kappa).$$

- (ii) *The natural map $\mathfrak{g} \otimes \mathcal{D}^0 \rightarrow \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})}$ is surjective.*

Proof. We first prove (i)(\Leftarrow). Write $D = \sum_{a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}} D_a \tilde{\pi}^{-a}$ and assume that $g \in \mathfrak{g} \otimes W(\kappa)$. Recall that the Baker–Campbell–Hausdorff formula takes the form $x \circ y = x + y + \frac{1}{2}[x, y]$. We have

$$\begin{aligned} g.D &= \sigma(g) \circ D \circ (-g) \\ &= \sigma(g) + D + (-g) + \frac{1}{2}[\sigma(g), D] + \frac{1}{2}[D, -g] + \frac{1}{2}[\sigma(g), -g] \\ &= \sigma(g) \circ (-g) + D - \frac{1}{2}[D, \sigma(g) + g] \\ &= \underbrace{\sigma(g) \circ (-g) + D_0 - \frac{1}{2}[D_0, \sigma(g) + g]}_{\in \mathfrak{g} \otimes W(\kappa)} + \sum_{a \in \mathbb{N} \setminus p\mathbb{N}} \underbrace{\left(D_a - \frac{1}{2}[D_a, \sigma(g) + g] \right)}_{\in \mathfrak{g} \otimes W(\kappa)} \tilde{\pi}^{-a} \in \mathfrak{g} \otimes \mathcal{D}^0. \end{aligned}$$

For $\mathfrak{g} = \mathbb{Z}/p\mathbb{Z}$, (i)(\Rightarrow) and (ii) follow from Lemma 3.3 since $\mathfrak{g} \otimes \mathcal{D}^0 = \mathcal{D}^0/p\mathcal{D}^0$, $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) = \mathfrak{F}^{\text{perf}}$, $\mathfrak{g} \otimes W(\kappa) = \kappa$, and $g.D = \wp(g) + D$. To show (i)(\Rightarrow) and (ii) for arbitrary \mathfrak{g} , we proceed by induction on the size of \mathfrak{g} as in the proof of Lemma 2.18. The case $\mathfrak{g} = 0$ is clear, so we assume that \mathfrak{g} is non-zero. Pick a subalgebra $\mathfrak{n} \subseteq Z(\mathfrak{g})$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$. We have already shown the claims for the abelian Lie algebra \mathfrak{n} , and the claims hold for the Lie algebra $\mathfrak{g}/\mathfrak{n}$ by the induction hypothesis.

We now prove (i)(\Rightarrow). Assume that $g.D \in \mathfrak{g} \otimes \mathcal{D}^0$. Let $\bar{g} \in (\mathfrak{g}/\mathfrak{n}) \otimes W(\mathfrak{F}^{\text{perf}})$ be the projection of g . By hypothesis, the projections of both D and $g.D$ belong to $(\mathfrak{g}/\mathfrak{n}) \otimes \mathcal{D}^0$, so \bar{g} belongs to

$(\mathfrak{g}/\mathfrak{n}) \otimes W(\kappa)$ by the induction hypothesis. Pick an arbitrary lift $\gamma \in \mathfrak{g} \otimes W(\kappa)$ of \bar{g} , and let $\delta := g - \gamma$, which belongs to $\mathfrak{n} \otimes W(\mathfrak{F}^{\text{perf}}) \subseteq Z(\mathfrak{g}) \otimes W(\mathfrak{F}^{\text{perf}})$. We have $g.D = (\gamma + \delta).(D + 0) = \gamma.D + \wp(\delta)$ by Lemma 2.16, and $g.D \in \mathfrak{g} \otimes \mathcal{D}^0$ by hypothesis. By the implication (\Leftarrow) proved above, $\gamma.D \in \mathfrak{g} \otimes \mathcal{D}^0$, so $\wp(\delta) = g.D - \gamma.D \in \mathfrak{g} \otimes \mathcal{D}^0$. Since $\delta \in \mathfrak{n} \otimes W(\mathfrak{F}^{\text{perf}})$, it follows that $\wp(\delta) \in \mathfrak{n} \otimes \mathcal{D}^0$. As $\mathfrak{n} \simeq \mathbb{Z}/p\mathbb{Z}$ satisfies (i) (\Rightarrow) , we conclude that $\delta \in \mathfrak{n} \otimes W(\kappa)$. Hence, $g = \gamma + \delta \in \mathfrak{g} \otimes W(\kappa)$.

Finally, we prove (ii). Consider an element $m \in \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ and let $\bar{m} \in (\mathfrak{g}/\mathfrak{n}) \otimes W(\mathfrak{F}^{\text{perf}})$ be its projection. By the induction hypothesis, there is an element $\bar{g} \in (\mathfrak{g}/\mathfrak{n}) \otimes W(\mathfrak{F}^{\text{perf}})$ such that the element $\bar{n} := \bar{g}.\bar{m}$ belongs to $(\mathfrak{g}/\mathfrak{n}) \otimes \mathcal{D}^0$. Choose lifts $g \in \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ and $n \in \mathfrak{g} \otimes \mathcal{D}^0$ of \bar{g} and \bar{n} , respectively. The element $g.m - n$ belongs to $\mathfrak{n} \otimes W(\mathfrak{F}^{\text{perf}})$. As $\mathfrak{n} \simeq \mathbb{Z}/p\mathbb{Z}$ satisfies (ii), there is an element $h \in \mathfrak{n} \otimes W(\mathfrak{F}^{\text{perf}})$ such that $i := h.(g.m - n) \in \mathfrak{n} \otimes \mathcal{D}^0$. By Lemma 2.16, we have $(g + h).m = g.m + h.0$ and $i = h.(g.m - n) = g.m - n + h.0$, so $(g + h).m = n + i \in \mathfrak{g} \otimes \mathcal{D}^0$. We have shown that the $(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)$ -orbit of m intersects $\mathfrak{g} \otimes \mathcal{D}^0$. \square

Remark 3.5. By Lemma 3.4 (i) (\Leftarrow) , the action of $(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)$ on $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ restricts to an action of $(\mathfrak{g} \otimes W(\kappa), \circ)$ on $\mathfrak{g} \otimes \mathcal{D}^0$. We denote by $\mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)}$ the set of orbits of $\mathfrak{g} \otimes \mathcal{D}^0$ under the action of $(\mathfrak{g} \otimes W(\kappa), \circ)$.

Theorem 3.6 (Local approximate fundamental domain). *There is a bijection*

$$\alpha^0: \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})} \xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)}$$

whose inverse is the map induced by the inclusion $\mathfrak{g} \otimes \mathcal{D}^0 \rightarrow \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$. Moreover, if $[D] = \alpha^0([m])$, then $\text{Stab}_{(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)}(m) \simeq \text{Stab}_{(\mathfrak{g} \otimes W(\kappa), \circ)}(D)$.

Proof. Lemma 3.4 (i) implies that the natural map $\mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)} \rightarrow \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})}$ is injective and that $\text{Stab}_{(\mathfrak{g} \otimes W(\kappa), \circ)}(D) = \text{Stab}_{(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)}(D) \simeq \text{Stab}_{(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)}(m)$ if $[D] = \alpha^0([m])$. Lemma 3.4 (ii) implies the surjectivity of this map. \square

We again extend the definition of the last jump like in Definition 2.23:

Definition 3.7. If $[D] \in \mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)}$ is the image of $[m] \in \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})}$ under α^0 , we define $\text{lastjump}([D]) := \text{lastjump}([m])$. Note that $\text{lastjump}([D]) = \text{lastjump}(D)$.

3.2. Ramification equations. In this subsection, we introduce for each $v > 0$ a property $J(v)$ on elements $D \in \mathfrak{g} \otimes \mathcal{D}^0$. This property will later be shown to coincide with the condition that $\text{lastjump}(D) < v$ (Theorem 3.20).

For any $b \in \mathbb{N} \setminus p\mathbb{N}$ and $v \in \mathbb{R}_{>0}$, we define

$$\mu_v(b) := \max(0, \lceil \log_p(v/b) \rceil) = \min\{k \geq 0 \mid bp^k \geq v\} = |\{k \geq 0 \mid bp^k < v\}| \quad (3.1)$$

so that $\mu_v(b) = 0 \Leftrightarrow b \geq v$, that $bp^{\mu_v(b)} \geq v$ for all v , and that $bp^{\mu_v(b)} < pv$ when $b < pv$. Note the following property of μ_v :

Lemma 3.8. *For any $v > 0$, we have*

$$\sum_{a \in \mathbb{N} \setminus p\mathbb{N}} \mu_v(a) = \lceil v \rceil - 1.$$

Proof. We have $\mu_v(a) = |\{k \geq 0 \mid ap^k < v\}|$. As every integer $0 < \gamma < v$ can be uniquely written as ap^k for some $a \in \mathbb{N} \setminus p\mathbb{N}$ and some $k \in \mathbb{Z}$, we have

$$\sum_{a \in \mathbb{N} \setminus p\mathbb{N}} \mu_v(a) = |\{\gamma \in \mathbb{N} \mid 0 < \gamma < v\}| = \lceil v \rceil - 1. \quad \square$$

If $n_1 \geq n_2$ are integers, we define

$$\eta(n_1, n_2) := \begin{cases} 1 & \text{if } n_1 > n_2 \\ \frac{1}{2} & \text{if } n_1 = n_2. \end{cases} \quad (3.2)$$

Definition 3.9. Let $v \in \mathbb{R}_{>0}$, and let $D \in \mathfrak{g} \otimes \mathcal{D}^0$. Write $D = \sum_{a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}} D_a \tilde{\pi}^{-a}$ with $D_a \in \mathfrak{g} \otimes W(\kappa)$. We say that D satisfies property $J(v)$ if the following equalities hold for all $b \in \mathbb{N} \setminus p\mathbb{N}$:

$$p^{\mu_v(b)} \sigma^{\mu_v(b)}(D_b) = -b^{-1} \sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}, \\ n_1 \geq n_2 \geq 0: \\ bp^{\mu_v(b)} = a_1 p^{n_1} + a_2 p^{n_2}, \\ a_1 p^{n_1} < v, a_2 p^{n_2} < v}} \eta(n_1, n_2) a_1 p^{n_1} [\sigma^{n_1}(D_{a_1}), \sigma^{n_2}(D_{a_2})], \quad (3.3)$$

$$0 = \sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}, \\ n \geq 0: \\ b = a_1 p^{n+i} + a_2, \\ a_1 p^n < v, a_2 p^n < v}} a_1 p^n [\sigma^{n+i}(D_{a_1}), D_{a_2}] \quad \text{for any } i > 0 \text{ such that } bp^{-i} \geq v. \quad (3.4)$$

(The condition $a_2 p^{n_1} < v$ in Equation (3.3) is stronger than $a_2 p^{n_2} < v$. This is not a typo!)

Remark 3.10. The sum in Equation (3.3) can be simplified by remarking that either $n_2 = n_1$ (in which case $\eta(n_1, n_2) = \frac{1}{2}$ and $n_2 \leq \mu_v(b)$), or $n_1 > n_2$, in which case n_2 necessarily equals $\mu_v(b)$ (comparing valuations in $bp^{\mu_v(b)} = a_1 p^{n_1} + a_2 p^{n_2}$), $\eta(n_1, n_2) = 1$, and $a_2 \equiv b \pmod{p^{n_1 - \mu_v(b)}}$.

Example 3.11. Assume that \mathfrak{g} is abelian. Then, the right-hand sides of Equations (3.3) and (3.4) vanish, so property $J(v)$ means that $p^{\mu_v(b)} D_b = 0$ for all $b \in \mathbb{N} \setminus p\mathbb{N}$. We retrieve a fact from class field theory: the p -part of the inertia group of the maximal abelian extension of \mathfrak{F} with $\text{lastjump} < v$ is isomorphic to $\prod_{b \in \mathbb{N} \setminus p\mathbb{N}} \mathbb{Z}/p^{\mu_v(b)} \mathbb{Z}$. (See for example [Gun26, Lemma 4.1].)

Remark 3.12. When $p\mathfrak{g} = 0$ (i.e., \mathfrak{g} is a Lie \mathbb{F}_p -algebra), the equations of Definition 3.9 take a simpler form, given in Corollary 6.1. These equations are easier to analyze, and considering this special case (detailed in Subsections 6.1 and 6.2) is recommended for a first reading.

Proposition 3.13. Let $v > 0$, let $D \in \mathfrak{g} \otimes \mathcal{D}^0$, and assume that D satisfies $J(v)$. Then:

- (i) For all $b \in \mathbb{N} \setminus p\mathbb{N}$, $p^{\mu_v(b)} D_b$ belongs to $[\mathfrak{g}, \mathfrak{g}] \otimes W(\kappa)$, and in particular to $Z(\mathfrak{g}) \otimes W(\kappa)$.
- (ii) For all $b \in \mathbb{N} \setminus p\mathbb{N}$, $p^{\mu_v(b)} D_b$ is a p -torsion element.
- (iii) For all $b \in \mathbb{N} \setminus p\mathbb{N}$ such that $b \geq 2v$, we have $D_b = 0$.

Proof.

- (i) Applying $\sigma^{-\mu_v(b)}$ to Equation (3.3) (σ is an automorphism of $W(\kappa)$), we can express $p^{\mu_v(b)} D_b$ as a sum of elements of $[\mathfrak{g}, \mathfrak{g}] \otimes W(\kappa)$.
- (ii) By Equation (3.3), checking that $p^{\mu_v(b)} D_b$ is a p -torsion element amounts to the vanishing of the following sum

$$\sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}, \\ n_1 \geq n_2 \geq 0: \\ bp^{\mu_v(b)} = a_1 p^{n_1} + a_2 p^{n_2}, \\ a_1 p^{n_1} < v, a_2 p^{n_2} < v}} \eta(n_1, n_2) a_1 p^{n_1+1} [\sigma^{n_1}(D_{a_1}), \sigma^{n_2}(D_{a_2})].$$

By (i), the elements $p^{\mu_v(a_1)} D_{a_1}$ and $p^{\mu_v(a_2)} D_{a_2}$ are central, and in particular the commutator $p^{n_1+1} [\sigma^{n_1}(D_{a_1}), \sigma^{n_2}(D_{a_2})]$ vanishes as soon as $n_1 + 1 \geq \mu_v(a_1)$ or $n_1 + 1 \geq \mu_v(a_2)$. Therefore, non-zero terms can only occur when $\mu_v(a_1) > n_1 + 1$ and $\mu_v(a_2) > n_1 + 1$, i.e., when $a_1 p^{n_1} < \frac{v}{p}$ and $a_2 p^{n_2} < \frac{v}{p}$. But then, the two inequalities $a_1 p^{n_1} + a_2 p^{n_2} < 2\frac{v}{p} < v$ and $bp^{\mu_v(b)} \geq v$ (by definition of $\mu_v(b)$) contradict the equality $a_1 p^{n_1} + a_2 p^{n_2} = bp^{\mu_v(b)}$, meaning that there are no non-zero terms in the sum.

- (iii) The inequality $b \geq 2v$ implies $\mu_v(b) = 0$. By Equation (3.3), we have

$$D_b = -b^{-1} \sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}, \\ n_1 \geq n_2 \geq 0: \\ b = a_1 p^{n_1} + a_2 p^{n_2}, \\ a_1 p^{n_1} < v, a_2 p^{n_2} < v}} \eta(n_1, n_2) a_1 p^{n_1} [\sigma^{n_1}(D_{a_1}), \sigma^{n_2}(D_{a_2})].$$

An equality $b = a_1 p^{n_1} + a_2 p^{n_2}$ with $a_1 p^{n_1}, a_2 p^{n_2} < v$ would imply $b < 2v$, which is not true. Hence, the sum is empty and $D_b = 0$. \square

3.3. The second fundamental domain. We define a second “fundamental domain” $\mathfrak{g} \otimes \mathcal{D}$, which is “lossy” (it essentially forgets about the unramified part and thus several G -extensions are mapped to the same element). However, it turns out that it retains enough information to determine the last jump of extensions (cf. Corollary 3.23). This second fundamental domain will be useful to establish our local–global principle and for counting.

Definition 3.14. We define the following free $W(\kappa)$ -submodule of $W(\mathfrak{F}) \subseteq W(\mathfrak{F}^{\text{perf}})$:

$$\mathcal{D} := \bigoplus_{a \in \mathbb{N} \setminus p\mathbb{N}} W(\kappa) \tilde{\pi}^{-a}$$

and we denote by $\text{pr}: \mathcal{D}^0 \rightarrow \mathcal{D}$ the natural projection, discarding the summand for $a = 0$.

The surjection $\text{pr}: \mathcal{D}^0 \rightarrow \mathcal{D}$ induces a surjection $\mathfrak{g} \otimes \mathcal{D}^0 \rightarrow \mathfrak{g} \otimes \mathcal{D}$, which we also denote by pr . Concretely, it maps an element $D_0 + \sum_{a \in \mathbb{N} \setminus p\mathbb{N}} D_a \tilde{\pi}^{-a}$ of $\mathfrak{g} \otimes \mathcal{D}^0$ to the element $\sum_{a \in \mathbb{N} \setminus p\mathbb{N}} D_a \tilde{\pi}^{-a}$. In particular, each fiber of pr has (finite) size $|\mathfrak{g} \otimes W(\kappa)|$.

Note that the variable D_0 does not appear in Equations (3.3) and (3.4). Therefore, whether an element $D \in \mathfrak{g} \otimes \mathcal{D}^0$ satisfies $J(v)$ only depends on its projection $\text{pr}(D) \in \mathfrak{g} \otimes \mathcal{D}$, and it makes sense to extend the definition of property $J(v)$ to $\mathfrak{g} \otimes \mathcal{D}$ as follows:

Definition 3.15. Let $v > 0$. We say that an element $D = \sum_{a \in \mathbb{N} \setminus p\mathbb{N}} D_a \tilde{\pi}^{-a}$ of $\mathfrak{g} \otimes \mathcal{D}$ satisfies property $J(v)$ if Equations (3.3) and (3.4) hold for all $b \in \mathbb{N} \setminus p\mathbb{N}$.

Proposition 3.16. *If $g \in \mathfrak{g} \otimes W(\kappa)$ and $D \in \mathfrak{g} \otimes \mathcal{D}^0$, then the element $\text{pr}(g.D) \in \mathcal{D}$ depends only on g and $\text{pr}(D)$. In other words, there is a (unique) action of $(\mathfrak{g} \otimes W(\kappa), \circ)$ on $\mathfrak{g} \otimes \mathcal{D}$ such that $\text{pr}(g.D) = g.\text{pr}(D)$ for all $D \in \mathfrak{g} \otimes \mathcal{D}^0$. Moreover:*

- (i) *This action is $W(\kappa)$ -linear, i.e., it is an action on the $W(\kappa)$ -module $\mathfrak{g} \otimes \mathcal{D}$;*
- (ii) *$\mathfrak{g} \otimes W(\kappa)$ acts trivially on $Z(\mathfrak{g}) \otimes \mathcal{D}$, so that orbits of central elements are of size 1;*
- (iii) *Let $v > 0$. For any $g \in \mathfrak{g} \otimes W(\kappa)$ and any $D \in \mathfrak{g} \otimes \mathcal{D}$ satisfying $J(v)$, the element $g.D$ satisfies $J(v)$.*

Proof. For any $D \in \mathfrak{g} \otimes \mathcal{D}^0$, the computation in the proof of Lemma 3.4 (i) (\Leftarrow) shows that

$$\text{pr}(g.D) = \sum_{a \in \mathbb{N} \setminus p\mathbb{N}} \left(D_a - \frac{1}{2} [D_a, \sigma(g) + g] \right) \tilde{\pi}^{-a}.$$

Since D_0 does not appear in this formula, $\text{pr}(g.D)$ only depends on g and $D' := \text{pr}(D)$. This dependency is given by the action of $(\mathfrak{g} \otimes W(\kappa), \circ)$ on elements $D' \in \mathfrak{g} \otimes \mathcal{D}$ via

$$g.D' := D' - \frac{1}{2} [D', \sigma(g) + g]. \quad (3.5)$$

That action is visibly $W(\kappa)$ -linear in D' , and the action on a central element $D' \in Z(\mathfrak{g}) \otimes \mathcal{D}$ is indeed trivial. With this definition, we have $\text{pr}(g.D) = g.\text{pr}(D)$.

It remains to check that $g.D$ satisfies $J(v)$ if $g \in \mathfrak{g} \otimes W(\kappa)$ and $D \in \mathfrak{g} \otimes \mathcal{D}$ satisfies $J(v)$. As illustrated by Equation (3.5), $g.D$ and D only differ by a commutator (hence an element of $Z(\mathfrak{g}) \otimes \mathcal{D}$). Thus, the invariance of Equation (3.4) is immediate. In Equation (3.3), the right-hand side is also left unchanged, so it suffices to prove that

$$p^{\mu_v(b)} D_b = p^{\mu_v(b)} \left(D_b - \frac{1}{2} [D_b, \sigma(g) + g] \right).$$

By Proposition 3.13 (ii), the element $p^{\mu_v(b)} D_b$ is central. Therefore, $p^{\mu_v(b)} [D_b, \sigma(g) + g] = 0$, which proves the claim. \square

Definition 3.17. We denote by $\mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}$ the set of orbits of $\mathfrak{g} \otimes \mathcal{D}$ under the $(\mathfrak{g} \otimes W(\kappa), \circ)$ -action of Proposition 3.16. For any $v > 0$, we say that an orbit $[D] \in \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}$ satisfies property $J(v)$ if D satisfies property $J(v)$ (by Proposition 3.16 (iii), this is independent of the choice of D).

Definition 3.18. We define the map $\alpha: \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})} \rightarrow \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}$ as the composition

$$\begin{array}{ccc} \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})} & \xrightarrow[\sim]{\alpha^0} & \mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)} \\ & \searrow \alpha & \downarrow \text{pr} \\ & & \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)} \end{array}$$

Lemma 3.19 (Twisting). *Let \mathfrak{n} be a Lie subalgebra of the center of \mathfrak{g} . For any $m \in \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ and $n \in \mathfrak{n} \otimes W(\mathfrak{F}^{\text{perf}})$, if*

$$\alpha([m]) = [D] \in \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)} \quad \text{and} \quad \alpha([n]) = [E] \in \mathfrak{n} \otimes \mathcal{D} //_{W(\kappa)},$$

then

$$\alpha([m+n]) = [D+E] \in \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}.$$

Proof. By definition of α , there are elements $g \in \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ and $h \in \mathfrak{n} \otimes W(\mathfrak{F}^{\text{perf}})$ such that $\text{pr}(g.m) = D$ and $\text{pr}(h.n) = E$. By Lemma 2.16, we have $\text{pr}((g+h).(m+n)) = \text{pr}(g.m + h.n) = \text{pr}(g.m) + \text{pr}(h.n) = D + E$, which belongs to $\mathfrak{g} \otimes \mathcal{D}$. So $\alpha([m+n]) = [D+E]$. \square

3.4. Determination of the last jump. The goal of this subsection is to deduce the following theorem from the results of [Abr98]:

Theorem 3.20. *Let $m \in \mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}})$ and $v \in \mathbb{R}_{>0}$. Then, the following are equivalent:*

- (a) $\text{lastjump}(m) < v$.
- (b) $\alpha^0([m])$ satisfies property $J(v)$.
- (c) $\alpha([m])$ satisfies property $J(v)$.

We briefly recall Abrashkin's results in his own notation. For a non-increasing list of integers

$$\underline{n} = (n_1, \dots, n_s) = (\underbrace{m_1, \dots, m_1}_{d_1}, \dots, \underbrace{m_k, \dots, m_k}_{d_k}) \in \mathbb{Z}^s$$

with $m_1 > \dots > m_k$, define the rational number (generalizing Equation (3.2))

$$\eta(\underline{n}) := \frac{1}{|\text{Stab}_{\mathfrak{S}_s}(\underline{n})|} = \frac{1}{d_1! \cdots d_k!}.$$

For any integer $N \geq 0$, any rational number $\gamma > 0$, and any element $D = \sum_{a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}} D_a \tilde{\pi}^{-a}$ of $\mathfrak{g} \otimes \mathcal{D}^0$ (with $D_a \in \mathfrak{g} \otimes W(\kappa)$), define the following element of $\mathfrak{g} \otimes W(\kappa)$:

$$\mathcal{F}_{\gamma, -N}(D) := \sum_{\substack{1 \leq s < p \\ \underline{a}, \underline{n}}} \eta(\underline{n}) a_1 p^{n_1} [\sigma^{n_1}(D_{a_1}), \dots, \sigma^{n_s}(D_{a_s})], \quad (3.6)$$

where the sum is over all lengths $s \in \{1, \dots, p-1\}$, all s -tuples $\underline{a} = (a_1, \dots, a_s) \in (\{0\} \cup \mathbb{N} \setminus p\mathbb{N})^s$, and all s -tuples $\underline{n} = (n_1, \dots, n_s) \in \mathbb{Z}^s$ satisfying $n_1 \geq 0$, $n_1 \geq \dots \geq n_s \geq -N$ and

$$a_1 p^{n_1} + \dots + a_s p^{n_s} = \gamma. \quad (3.7)$$

The main result from [Abr98] implies the following formula for the last jump (valid for all Lie \mathbb{Z}_p -algebras \mathfrak{g} of nilpotency class $< p$):

Theorem 3.21 (Abrashkin's theorem). *Let $D \in \mathfrak{g} \otimes \mathcal{D}^0$, and assume that the corresponding G -extension of \mathfrak{F} is a field. Then:*

$$\text{lastjump}(D) = \sup \left(\{0\} \cup \{ \gamma > 0 \mid \exists N_0 \geq 0, \forall N \geq N_0, \mathcal{F}_{\gamma, -N}(D) \neq 0 \} \right).$$

Proof. Consider a surjective continuous group homomorphism $f: \Gamma_{\mathfrak{F}} \rightarrow G$ in the class $\text{orb}^{-1}([D])$. For every $v > 0$, let \mathfrak{g}^v be the Lie subalgebra of \mathfrak{g} corresponding to the image in G of the ramification subgroup $\Gamma_{\mathfrak{F}}^v$ under f . By definition, we have $\text{lastjump}(D) = \inf \{v > 0 \mid \mathfrak{g}^v = 0\} = \sup (\{0\} \cup \{v > 0 \mid \mathfrak{g}^v \neq 0\})$. For each $v > 0$, by [Abr23, Theorem 3.1] (which is the ‘‘covariant’’ version of [Abr98, Theorem B]), there is an integer N such that \mathfrak{g}^v is the smallest ideal of \mathfrak{g} whose extension of scalars $\mathfrak{g} \otimes W(\kappa)$ contains $\mathcal{F}_{\gamma, -N}(D)$ for all $\gamma \geq v$. Therefore, the condition

$\mathfrak{g}^v = 0$ is equivalent to the vanishing of $\mathcal{F}_{\gamma, -N}(D)$ for all $\gamma \geq v$ and N large enough. The result follows directly. \square

We now specialize the definition of $\mathcal{F}_{\gamma, -N}(D)$ (Equation (3.6)) to our situation. Note that Equation (3.7) can only hold if $\gamma \in \mathbb{N}[\frac{1}{p}]$. Hence, for $\gamma \notin \mathbb{N}[\frac{1}{p}]$, we always have $\mathcal{F}_{\gamma, -N}(D) = 0$. We now assume that $\gamma = bp^m$ for some $b \in \mathbb{N} \setminus p\mathbb{N}$ and $m \in \mathbb{Z}$. In our case, since \mathfrak{g} has nilpotency class ≤ 2 , we need only to consider lengths $s \leq 2$ as all commutators involving more elements vanish.

- For $s = 1$, the equality $a_1 p^{n_1} = \gamma$ (with $a_1 \in \mathbb{N} \setminus p\mathbb{N}$, $n_1 \geq 0$) can only happen if γ is an integer ($m \geq 0$). In that case, $a_1 = b$ and $n_1 = m$, and the corresponding term is $bp^m \sigma^m(D_b)$.
- For $s = 2$, the equality $a_1 p^{n_1} + a_2 p^{n_2} = \gamma$ (with $n_1 \geq 0$) implies:
 - If γ is an integer ($m \geq 0$), then either $a_2 = 0$ (and then $a_1 = b$, $n_1 = m$), or $a_2 \neq 0$ and $n_2 \geq 0$.
 - If γ is not an integer ($m < 0$), then $n_2 < 0$ and $a_2 \neq 0$. In particular, $n_2 < n_1$ and thus $\eta(n_1, n_2) = 1$. Comparing valuations shows that $n_2 = m$ (there are no terms if $N \leq -m$).

Thus, we have the following expression for $\mathcal{F}_{bp^m, -N}(D)$ when $b \in \mathbb{N} \setminus p\mathbb{N}$ and $m \geq 0$:

$$\begin{aligned} \mathcal{F}_{bp^m, -N}(D) &= bp^m \sigma^m(D_b) && \text{(for } s = 1) \\ &+ \sum_{n=-N}^m \eta(m, n) bp^m [\sigma^m(D_b), \sigma^n(D_0)] && \text{(for } s = 2, a_2 = 0) \\ &+ \sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}, \\ n_1 \geq n_2 \geq 0: \\ bp^m = a_1 p^{n_1} + a_2 p^{n_2}}} \eta(n_1, n_2) a_1 p^{n_1} [\sigma^{n_1}(D_{a_1}), \sigma^{n_2}(D_{a_2})] && \text{(for } s = 2, a_2 \neq 0) \end{aligned}$$

and the following expression for $\mathcal{F}_{bp^m, -N}(D)$ when $b \in \mathbb{N} \setminus p\mathbb{N}$ and $m < 0$, assuming $N \geq -m$:

$$\mathcal{F}_{bp^m, -N}(D) = \sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}, \\ n_1 \geq 0: \\ bp^m = a_1 p^{n_1} + a_2 p^m}} a_1 p^{n_1} [\sigma^{n_1}(D_{a_1}), \sigma^m(D_{a_2})].$$

Proposition 3.22. *Let $D \in \mathfrak{g} \otimes \mathcal{D}^0$. Then, D satisfies property $J(v)$ if and only if, for all $b \in \mathbb{N} \setminus p\mathbb{N}$ and $m \in \mathbb{Z}$ such that $bp^m \geq v$, and for every sufficiently large N , we have $\mathcal{F}_{bp^m, -N}(D) = 0$.*

Proof. For every $b \in \mathbb{N} \setminus p\mathbb{N}$, both property $J(v)$ and the equality $\mathcal{F}_{bp^{\mu_v(b)}, -N}(D) = 0$ independently imply that $p^{\mu_v(b)} D_b \in [\mathfrak{g}, \mathfrak{g}] \otimes W(\kappa)$ (see Proposition 3.13 (i) and the expression for $\mathcal{F}_{bp^m, -N}(D)$ above). We may hence prove the desired equivalence under the assumption that $p^{\mu_v(b)} D_b \in [\mathfrak{g}, \mathfrak{g}] \otimes W(\kappa)$ for all $b \in \mathbb{N} \setminus p\mathbb{N}$. In particular, $p^m D_b \in Z(\mathfrak{g}) \otimes W(\kappa)$ for all $m \geq 0$ such that $bp^m \geq v$.

Under this assumption, every term in $\mathcal{F}_{bp^m, -N}(D)$ with $a_1 p^{n_1} \geq v$ or $a_2 p^{n_2} \geq v$ can be omitted, as the corresponding commutator vanishes once multiplied by p^{n_1} . Thus, the vanishing of $\mathcal{F}_{bp^m, -N}(D)$ for every $m < 0$ such that $bp^m \geq v$ is equivalent to Equation (3.4) (with $i := -m$). Similarly, for all $m \geq \mu_v(b)$, the sum over n in $\mathcal{F}_{bp^m, -N}(D)$ (the one involving D_0) can be omitted, and the vanishing of $\mathcal{F}_{bp^{\mu_v(b)}, -N}(D)$ is equivalent to Equation (3.3).

All that is left to do is to check that property $J(v)$ implies the vanishing of $\mathcal{F}_{bp^m, -N}(D)$ when $m > \mu_v(b)$. In that case, the inequality $bp^m \geq pv > 2v$ makes it impossible to simultaneously have $a_1 p^{n_1} + a_2 p^{n_2} = bp^m$ and $a_1 p^{n_1}, a_2 p^{n_2} < v$. Thus, all terms in $\mathcal{F}_{bp^m, -N}(D)$ vanish except for $bp^m \sigma^m(D_b)$, but that term also vanishes because $m > \mu_v(b)$ and $p^{\mu_v(b)} D_b$ is p -torsion by Proposition 3.13 (ii). \square

Finally, we prove Theorem 3.20:

Proof of Theorem 3.20. The equivalence (b) \Leftrightarrow (c) is clear (see Definitions 3.15 and 3.17)

Pick a $D \in \mathfrak{g} \otimes \mathcal{D}^0$ such that $\alpha^0([m]) = [D]$. By definition, m and D belong to the same $(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)$ -orbit, and thus $\text{lastjump}(m) = \text{lastjump}(D)$ (they define the same extension!). Pick a $g \in \mathfrak{g} \otimes W(\mathfrak{F}^{\text{alg}})$ such that $\wp(g) = D$, and consider the continuous group homomorphism $f: \Gamma_{\mathfrak{F}} \rightarrow G$ defined by $\tau \mapsto (-g) \circ \tau(g)$.

If f is surjective, the corresponding G -extension of \mathfrak{F} is a field, and the result follows from Theorem 3.21 and Proposition 3.22 (recall that $\mathcal{F}_{\gamma, -N}(D)$ always vanishes if γ is not of the form bp^m for $b \in \mathbb{N} \setminus p\mathbb{N}$ and $m \in \mathbb{Z}$).

Assume now that f is not surjective. The image of f is then a certain subgroup $H \subset G$, corresponding to a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then, the G -extension associated to f is not a field, but the corresponding H -extension is a field. In that case, by Theorem 2.19 applied to \mathfrak{h} , there is a $D' \in \mathfrak{h} \otimes \mathcal{D}^0$ and a $g' \in \mathfrak{h} \otimes W(\mathfrak{F}^{\text{alg}})$ such that $D' = \wp(g')$, and $f(\tau) = (-g') \circ \tau(g')$. By the surjective case above (applied to the Lie algebra \mathfrak{h}), we have

$$\text{lastjump}(D') < v \quad \iff \quad D' \text{ satisfies property } J(v).$$

Since D and D' induce the same element $[f]$ of $H^1(\Gamma_{\mathfrak{F}}, G)$, we have $\text{lastjump}(D) = \text{lastjump}(D')$, and D and D' are in the same $(\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}), \circ)$ -orbit by Theorem 2.19. By Remark 3.5, D and D' are then in the same $(\mathfrak{g} \otimes W(\kappa), \circ)$ -orbit, and by Proposition 3.16 (iii) we have

$$D \text{ satisfies property } J(v) \quad \iff \quad D' \text{ satisfies property } J(v).$$

Assembling everything together, we obtain (a) \Leftrightarrow (b). \square

Corollary 3.23 (Irrelevance of D_0). *Let $m \in \mathfrak{g} \otimes \mathcal{D}$. By Theorem 3.20, all elements $m' \in \mathfrak{g} \otimes \mathcal{D}^0$ such that $\text{pr}(m') = m$ have the same last jump $\text{lastjump}(m')$, namely*

$$\inf\left(\{0\} \cup \{v > 0 \mid m \text{ satisfies property } J(v)\}\right).$$

Definition 3.24. If $m \in \mathfrak{g} \otimes \mathcal{D}$, we define $\text{lastjump}(m)$ to be the common value of $\text{lastjump}(m')$ for all $m' \in \text{pr}^{-1}(m)$ (cf. Corollary 3.23), and we let $\text{lastjump}([m]) := \text{lastjump}(m)$ where $[m] \in \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}$ is the $(\mathfrak{g} \otimes W(\kappa), \circ)$ -orbit of m (this is well-defined by Proposition 3.16 (iii)).

4. COUNTING LOCAL EXTENSIONS

The setting and notations are identical to those of Section 3. This section deals with obtaining estimates of the number

$$\sum_{\substack{K \in \text{ÉtExt}(G, \mathfrak{F}): \\ \text{lastjump}(K) < v}} \frac{1}{|\text{Aut}(K)|}$$

for various values of v , thus describing the distribution of last jumps of G -extensions of \mathfrak{F} . The starting point is the following observation:

Lemma 4.1. *Let $v > 0$. Using the notation introduced in Definition 3.24, we have the equality*

$$\sum_{\substack{K \in \text{ÉtExt}(G, \mathfrak{F}): \\ \text{lastjump}(K) < v}} \frac{1}{|\text{Aut}(K)|} = |\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < v\}|,$$

as well as the analogous equality if the condition $< v$ is replaced by $= v$.

Proof. We focus on the first claim, as the second claim directly follows from it. Using the three (stabilizer-preserving) bijections of Lemma 2.1, Theorem 2.19 and Theorem 3.6 and the notation of Definition 3.7, the left-hand side becomes

$$\begin{aligned} \sum_{\substack{K \in \text{ÉtExt}(G, \mathfrak{F}): \\ \text{lastjump}(K) < v}} \frac{1}{|\text{Aut}(K)|} &= \sum_{\substack{[D] \in \mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)}: \\ \text{lastjump}([D]) < v}} \frac{1}{|\text{Stab}_{(\mathfrak{g} \otimes W(\kappa), \circ)}(m)|} \\ &= \frac{|\{D \in \mathfrak{g} \otimes \mathcal{D}^0 \mid \text{lastjump}(D) < v\}|}{|\mathfrak{g} \otimes W(\kappa)|} \quad \text{by the orbit-stabilizer theorem.} \end{aligned}$$

Recalling Definition 3.24 and the fact that the projection map $\text{pr}: \mathfrak{g} \otimes \mathcal{D}^0 \rightarrow \mathfrak{g} \otimes \mathcal{D}$ is $|\mathfrak{g} \otimes W(\kappa)|$ -to-one, we see that this equals $|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < v\}|$ as claimed. \square

Let

$$r := \dim_{\mathbb{F}_p}(\mathfrak{g}[p]) \quad \text{and} \quad M := \begin{cases} 1 & \text{if } \mathfrak{g}[p] \text{ is abelian,} \\ 1 + p^{-1} & \text{otherwise,} \end{cases}$$

as in Theorem 1.2 for $G = (\mathfrak{g}, \circ)$. (Corollary 4.8 below gives an interpretation of M as the largest possible last jump smaller than 2 for G -extensions of \mathfrak{F} .)

The main results of this section are the following estimates for the number of elements of $\mathfrak{g} \otimes \mathcal{D}$ with bounded last jump (this is motivated by Lemma 4.1):

Theorem 4.2. *For some (explicit) constant $E \geq 1$ (which does not depend on κ), the following estimates hold:*

- (a) $|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) = 0\}| = |\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < 1\}| = 1.$
- (b) $|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) \leq M\}| = |\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < 2\}| = |\kappa|^r.$
- (c) *For any $l \in \mathbb{Z}_{\geq 0}$, we have $|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < l + 1\}| \leq E^l |\kappa|^{rl}.$*
- (d) *Assume that $\mathfrak{g}[p]$ is non-abelian. Then:*
 - (i) *for all integers $l \geq p$, we have $|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < l + 1\}| \leq E^l |\kappa|^{rl-1}.$*
 - (ii) *for all $l \in \{1, \dots, p-1\}$, we have*

$$|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < l(1 + p^{-1})\}| = \mathcal{O}\left(E^l |\kappa|^{rl-1}\right)$$

where the implied constant in the \mathcal{O} -estimate depends only on $\mathfrak{g}[p]$.

Points (a) and (b) follow from Lemma 4.5 and Corollary 4.8 respectively. The proof of points (c) and (d) will be the goal of Subsection 4.3.

Theorem 4.2 is later used for the global results of Section 5, where it is applied at each place P of the global function field $\mathbb{F}_q(T)$ by letting \mathfrak{F} be its completion at P and π be a uniformizer.

4.1. Smallness criteria for the last jump. We use Theorem 3.20 to characterize elements $D = \sum_{a \in \mathbb{N} \setminus p\mathbb{N}} D_a \tilde{\pi}^{-a} \in \mathfrak{g} \otimes \mathcal{D}$ whose last jump is “small”:

Corollary 4.3 (Small v). *Let $v \leq p$. We have $\text{lastjump}(D) < v$ if and only if*

$$p\sigma(D_b) = -(2b)^{-1} \sum_{\substack{1 \leq a_1, a_2 < v: \\ bp = a_1 + a_2}} a_1 [D_{a_1}, D_{a_2}] \quad \text{for all } b \in \mathbb{N} \setminus p\mathbb{N} \text{ with } b < v, \quad (4.1)$$

$$D_b = -(2b)^{-1} \sum_{\substack{1 \leq a_1, a_2 < v: \\ b = a_1 + a_2}} a_1 [D_{a_1}, D_{a_2}] \quad \text{for all } b \in \mathbb{N} \setminus p\mathbb{N} \text{ with } b \geq v, \quad (4.2)$$

$$0 = [\sigma^i D_{\lfloor v \rfloor}, D_a] \quad \text{for all } 1 \leq a < v \text{ and } i > 0 \text{ with } ap^{-i} \geq v - \lfloor v \rfloor \text{ if } v \text{ is not an integer.} \quad (4.3)$$

Proof. We apply Theorem 3.20. Here, $\mu_v(b) = \mathbb{1}_{b < v}$. In all summands in Equations (3.3) and (3.4), the conditions of the form $ap^n < v$ are equivalent to $n = 0$, $a < v$. Equations (4.1) and (4.2) thus correspond to Equation (3.3) (note that $\eta(n_1, n_2) = \frac{1}{2}$ as $n_1 = n_2 = 0$), and Equation (4.3) corresponds to Equation (3.4). (For $i > 0$, any $b \geq vp^i$ can be written in at most one way as $b = a_1 p^i + a_2$ with $1 \leq a_1, a_2 < v \leq p$, namely a_2 must be the remainder of b modulo p , and a_1 must be $\lfloor v \rfloor$. Hence, at most one summand appears in each instance of Equation (3.4). If v is an integer, then $a_1 = \lfloor v \rfloor = v$ is not strictly smaller than v .) \square

Corollary 4.4. *Let $v \leq p$. Given elements $D_a \in \mathfrak{g} \otimes W(\kappa)$ for all $a < v$, there is at most one choice of the remaining elements D_a for $a \geq v$ such that the element $D := \sum D_a \tilde{\pi}^{-a}$ satisfies $\text{lastjump}(D) < v$.*

Proof. This follows immediately from Equation (4.2). \square

We obtain the following characterization of unramified extensions, which directly implies Theorem 4.2 (a):

Lemma 4.5 (Unramified extensions). *The following are equivalent:*

- (i) $\text{lastjump}(D) < 1$.
- (ii) $D_a = 0$ for all $a \in \mathbb{N} \setminus p\mathbb{N}$.
- (iii) $\text{lastjump}(D) = 0$.

Proof. The implication (iii) \Rightarrow (i) is obvious. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) follow from Corollary 4.3 with $v = 1$ and $v \rightarrow 0^+$, respectively. \square

Proposition 4.6. *For any $m \geq 0$, the following are equivalent:*

- (i) $\text{lastjump}(D) < 1 + p^{-m}$
- (ii) $\text{lastjump}(D) \leq 1 + p^{-(m+1)}$
- (iii) $D_a = 0$ for all $a \geq 2$, $pD_1 = 0$, and $[\sigma^i D_1, D_1] = 0$ for all $0 < i \leq m$.

Proof. Apply Corollary 4.3. For any $v \in (1, 2]$, the inequalities $1 \leq a_1, a_2 < v$ imply that $a_1 = a_2 = 1$, and in particular commutators $[D_{a_1}, D_{a_2}]$ vanish. Therefore, Equation (4.1) is equivalent to $pD_1 = 0$, and Equation (4.2) is equivalent to $D_b = 0$ for $b \geq 2$. Finally, Equation (4.3) amounts to $[\sigma^i D_1, D_1] = 0$ for all $i > 0$ such that $p^{-i} \geq v - 1$. Thus, for any $v \in (1 + p^{-(m+1)}, 1 + p^{-m}]$, (iii) is equivalent to $\text{lastjump}(D) < v$, yielding the result. \square

Remark 4.7. Let $m \geq 0$. For any element $D \in \mathfrak{g} \otimes \mathcal{D}$, condition (iii) of Proposition 4.6 means that D is of the form $D_1 \tilde{\pi}^{-1}$ where D_1 is an element of $\mathfrak{g}[p] \otimes W(\kappa) = \mathfrak{g}[p] \otimes_{\mathbb{F}_p} \kappa$ whose projection x to $(\mathfrak{g}[p]/Z(\mathfrak{g}[p])) \otimes_{\mathbb{F}_p} \kappa$ belongs to the set

$$A_m(\kappa) := \{x \in (\mathfrak{g}[p]/Z(\mathfrak{g}[p])) \otimes \kappa \mid [\sigma^i(x), x] = 0 \text{ for all } 0 < i \leq m\}.$$

In particular, Proposition 4.6 implies that

$$|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < 1 + p^{-m}\}| = \underbrace{|A_m(\kappa)|}_{\text{choices of } x} \cdot \underbrace{|Z(\mathfrak{g}[p]) \otimes \kappa|}_{\text{choices of } D_1 \text{ once } x \text{ is fixed}}.$$

From Proposition 4.6, we deduce the following characterization, which directly implies Theorem 4.2 (b) (we have $|\mathfrak{g}[p] \otimes_{\mathbb{Z}_p} W(\kappa)| = |\mathfrak{g}[p] \otimes_{\mathbb{F}_p} \kappa| = |\kappa|^r$).

Corollary 4.8 (Slightly ramified extensions). *The following are equivalent:*

- (i) $\text{lastjump}(D) < 2$.
- (ii) $\text{lastjump}(D) \leq M$.
- (iii) $D_a = 0$ for all $a \geq 2$, and $pD_1 = 0$.

Proof. If $\mathfrak{g}[p]$ is non-abelian, this is simply the case $m = 0$ of Proposition 4.6. Assume now that $\mathfrak{g}[p]$ is abelian. Then, when $pD_1 = 0$, all commutators $[\sigma^i D_1, D_1]$ vanish, making condition (iii) of Proposition 4.6 independent of the value of m (it matches condition (iii) here). It remains only to observe that $\text{lastjump}(D) < 2$ implies (iii) and hence implies $\text{lastjump}(D) < 1 + p^{-m}$ for all $m \geq 0$, meaning that $\text{lastjump}(D) \leq 1$. \square

Finally, we generalize the implication (i) \Rightarrow (iii) of Proposition 4.6 to slightly larger values of the last jump:

Proposition 4.9. *Let $1 \leq l \leq p - 1$ and $m \geq 1$.*

- (i) *If $\text{lastjump}(D) < l + p^{-m}$, then $[\sigma^i D_l, D_a] = 0$ for all $1 \leq i \leq m$ and $1 \leq a \leq l$.*
- (ii) *If $\text{lastjump}(D) < l + lp^{-m}$, then $[\sigma^i D_l, D_a] = 0$ for all $1 \leq i \leq m - 1$ and $1 \leq a \leq l - 1$ and $[\sigma^i D_l, D_l] = 0$ for all $1 \leq i \leq m$.*

Proof.

- (i) For any $1 \leq i \leq m$ and $1 \leq a \leq l$, we have $ap^{-i} \geq p^{-m}$. Hence, Equation (4.3) implies $[\sigma^i D_l, D_a] = 0$.
- (ii) This follows in the same way since $ap^{-i} \geq lp^{-m}$ for $1 \leq i \leq m - 1$ and $1 \leq a \leq l - 1$ and since $lp^{-i} \geq lp^{-m}$ for $1 \leq i \leq m$. \square

4.2. Equations in Witt vectors. Before we start proving points (c) and (d) of Theorem 4.2, we advise the reader to first read the proof in the case where \mathfrak{g} has exponent p , which is given in Proposition 6.2: this special case is much easier to deal with, and gives a simple illustration of the rough idea of our general proof — namely that fixing D_b for the smallest few values of b essentially determines D_b for all b .

Let $r := \dim_{\mathbb{F}_p}(\mathfrak{g}[p])$. The finite \mathbb{Z}_p -module \mathfrak{g} decomposes into a product

$$\mathfrak{g} \simeq \prod_{i=1}^r \mathbb{Z}/p^{n_i}\mathbb{Z} \quad \text{as } \mathbb{Z}_p\text{-modules.}$$

We have $\mathfrak{g} \otimes W(\kappa) \simeq \prod_i W_{n_i}(\kappa)$ as \mathbb{Z}_p -modules (see also Lemma 3.1). If X is an element of $\mathfrak{g} \otimes W(\kappa)$, $1 \leq i \leq r$, and $0 \leq j < n_i$, we denote by $X^{(ij)} \in \kappa$ the j -th coordinate of the i -th Witt vector (in $W_{n_i}(\kappa)$) associated to X .

Let $v \in \mathbb{R}_{>0}$. According to Theorem 3.20, equations characterizing elements $D \in \mathfrak{g} \otimes \mathcal{D}$ with $\text{lastjump}(D) < v$ are given in Definition 3.9. We show below that these are polynomial equations¹ in the indeterminates $D_a^{(ij)} \in \kappa$, where the triple (a, i, j) belongs to the countable set

$$\Omega := \{(a, i, j) \mid a \in \mathbb{N} \setminus p\mathbb{N}, 1 \leq i \leq r, 0 \leq j < n_i\}.$$

Moreover, these equations do not depend on anything besides the Lie algebra \mathfrak{g} and the real number v (they depend neither on \mathfrak{F} nor on κ , and have coefficients in \mathbb{F}_p).

Consider the polynomial ring in countably many variables

$$\mathcal{R} := \mathbb{F}_p[(D_a^{(ij)})_{(a,i,j) \in \Omega}]$$

and its perfect closure

$$\mathcal{R}^{\text{perf}} = \mathbb{F}_p[((D_a^{(ij)})^{p^{-\infty}})_{(a,i,j) \in \Omega}].$$

Since $\prod_i W_{n_i}(\mathcal{R})$ is the image of the Lie algebra $\mathfrak{g} \otimes W(\mathcal{R})$ in $\mathfrak{g} \otimes W(\mathcal{R}^{\text{perf}}) = \prod_i W_{n_i}(\mathcal{R}^{\text{perf}})$, it naturally inherits a Lie algebra structure. Define D_a to be the element of $\prod_i W_{n_i}(\mathcal{R})$ for which the j -th coordinate of the i -th Witt vector is the indeterminate $D_a^{(ij)}$. Then, each side of the Equations (3.3) and (3.4) defining property $J(v)$ (for given b and i) can be interpreted as an element of $\prod_i W_{n_i}(\mathcal{R})$. Equating the coordinates of these elements for all admissible values of b and i , we obtain (polynomial) equations in the indeterminates $D_a^{(ij)}$. We let \mathcal{I}_v be the ideal of \mathcal{R} associated to these equations.

By construction, elements $D \in \mathfrak{g} \otimes \mathcal{D}$ with $\text{lastjump}(D) < v$ are in one-to-one correspondence with the solutions to these polynomial equations in κ^Ω , i.e., with the κ -points of the affine \mathbb{F}_p -scheme $\mathbb{J}_v := \text{Spec}(\mathcal{R}/\mathcal{I}_v)$. (That all but finitely many coordinates vanish actually follows from the equations, cf. Proposition 3.13 (iii).)

The following remark is not used in this paper, but it suggests a moduli space interpretation:

Remark 4.10. Let $\mathbb{J}_v^0 := \mathbb{A}_{\mathbb{F}_p}^{\log_p|\mathfrak{g}|} \times \mathbb{J}_v$ (reincorporating the variables $D_0^{(ij)}$). By Theorem 3.20, there is a bijection $\mathbb{J}_v^0(\kappa) \simeq \{D \in \mathfrak{g} \otimes \mathcal{D}^0 \mid \text{lastjump}(D) < v\}$. Up to some action, the space \mathbb{J}_v^0 thus parametrizes ramified G -covers of the one-dimensional infinitesimal neighborhood of a single point where the “depth” of wild ramification is $< v$. This bears some analogy to Hurwitz spaces, which parametrize tame covers of the line whose branch divisor has a fixed degree.

Grading. We define a $\mathbb{Q}_{\geq 0}$ -grading on the \mathbb{F}_p -algebra $\mathcal{R}^{\text{perf}}$ by assigning the degree $p^{-\mu_v(a)+j}$ to the generator $D_a^{(ij)}$. Note that $\mu_v(a)$ is bounded, so the degrees of monomials in \mathcal{R} form a discrete (hence well-founded) subset of $\mathbb{R}_{\geq 0}$. The following general remark shows how the grading on $\mathcal{R}^{\text{perf}}$ induces a partial grading on $\mathfrak{g} \otimes W(\mathcal{R}^{\text{perf}})$, for which $D_a \in \mathfrak{g} \otimes W(\mathcal{R}^{\text{perf}})$ is homogeneous of degree $p^{-\mu_v(a)}$:

¹As equations over the Witt vectors, these equations are not polynomial as they involve the absolute Frobenius endomorphism σ , which is why we rephrase them as equations over κ . They are, however, “algebraic difference equations” (in the sense of [Bél09, SV22, HHYZ24]) over the difference ring $(W(\kappa), \sigma)$.

Remark 4.11. Let $R = \bigoplus_{d \in \mathbb{Q}_{\geq 0}} R_d$ be a $\mathbb{Q}_{\geq 0}$ -graded ring of characteristic p . For any $d \in \mathbb{Q}_{\geq 0}$, let $V_d \subseteq W(R)$ be the set of Witt vectors over R whose j -th coordinate is homogeneous of degree dp^j for all j , i.e.:

$$V_d := \{(x_0, x_1, \dots) \in W(R) \mid x_j \in R_{dp^j} \text{ for all } j \geq 0\}.$$

Looking at the definition of addition and multiplication of Witt vectors, one verifies that each V_d is a \mathbb{Z}_p -submodule of $W(R)$, that if $(x_0, x_1, \dots) \in \bigoplus_{d' \leq d} V_{d'}$, then $x_j \in \bigoplus_{d' \leq d} R_{d'p^j}$ for all $j \geq 0$, and that $V_d \cdot V_e \subseteq V_{d+e}$. Moreover, if σ is the Frobenius endomorphism of R , then $\sigma(V_d) \subseteq V_{dp}$. If \mathfrak{g} is a Lie \mathbb{Z}_p -algebra, we obtain \mathbb{Z}_p -submodules $\mathfrak{g} \otimes V_d$ of $\mathfrak{g} \otimes W(R)$ satisfying $[\mathfrak{g} \otimes V_d, \mathfrak{g} \otimes V_e] \subseteq \mathfrak{g} \otimes V_{d+e}$. We call the elements of V_d or of $\mathfrak{g} \otimes V_d$ *homogeneous of degree d* .

4.3. Bounding the number of extensions with large last jump.

Proving Theorem 4.2 (c). Fix an integer $l \geq 0$, and define the following finite subset of Ω :

$$\Omega_l^\bullet := \{(a, i, j) \in \Omega \mid n_i - \mu_{l+1}(a) \leq j\},$$

whose cardinality satisfies

$$|\Omega_l^\bullet| = \sum_{a \in \mathbb{N} \setminus p\mathbb{N}} \sum_{i=1}^r \min(n_i, \mu_{l+1}(a)) \leq r \sum_{a \in \mathbb{N} \setminus p\mathbb{N}} \mu_{l+1}(a) \stackrel{\text{Lemma 3.8}}{=} rl. \quad (4.4)$$

We shall show that, if $D \in \mathfrak{g} \otimes \mathcal{D}$ is such that $\text{lastjump}(D) < l + 1$, then the coordinates $D_a^{(ij)}$ for triples $(a, i, j) \in \Omega_l^\bullet$ determine D up to at most $|G|^{2l}$ choices. To this end, we introduce the following polynomial ring in $|\Omega_l^\bullet|$ variables:

$$\mathcal{R}_l^\bullet := \mathbb{F}_p[(D_a^{(ij)})_{(a,i,j) \in \Omega_l^\bullet}].$$

Proposition 4.12. *The k -algebra $\mathcal{R}/\mathcal{I}_{l+1}$ is a finite \mathcal{R}_l^\bullet -module of rank at most $|G|^{2l}$.*

Proof. We are going to show that every polynomial in \mathcal{R} is congruent, modulo \mathcal{I}_{l+1} , to a linear combination with coefficients in \mathcal{R}_l^\bullet of monomials of the form

$$\prod_{(a,i,j) \in \Omega \setminus \Omega_l^\bullet} (D_a^{(ij)})^{e_{aij}} \quad \text{where } 0 \leq e_{aij} < p^{2\mu_{l+1}(a)}. \quad (4.5)$$

As the number of such monomials is $\prod_{(a,i,j) \in \Omega \setminus \Omega_l^\bullet} p^{2\mu_{l+1}(a)}$, which by Lemma 3.8 is at most $p^{2l \sum_i n_i} = |G|^{2l}$, this implies the result.

Recall that D_a is homogeneous of degree $p^{-\mu_{l+1}(a)}$ for any a , and recall the properties listed in Remark 4.11. We see that the left-hand side $p^{\mu_{l+1}(b)} \sigma^{\mu_{l+1}(b)}(D_b)$ of Equation (3.3) is homogeneous of degree $p^{\mu_{l+1}(b)} \cdot p^{-\mu_{l+1}(b)} = 1$. (The multiplication by $p^{\mu_{l+1}(b)}$ does not change the degree. The Frobenius homomorphism $\sigma^{\mu_{l+1}(b)}$ multiplies the degree by $p^{\mu_{l+1}(b)}$.) The right-hand side is a sum of terms which are homogeneous of degree at most $\frac{2}{p} < 1$. ($\sigma^{n_1}(D_{a_1})$ is homogeneous of degree $p^{n_1} \cdot p^{-\mu_{l+1}(a_1)}$, which is $\leq \frac{1}{p}$ as $a_1 p^{n_1} < l + 1$, and similarly for $\sigma^{n_2}(D_{a_2})$, so each summand is homogeneous of degree at most $\frac{2}{p}$.)

Assume now that (a, i, j) belongs to $\Omega \setminus \Omega_l^\bullet$, i.e., that $j + \mu_{l+1}(a) < n_i$, and consider the $(i, j + \mu_{l+1}(a))$ -coordinate of Equation (3.3) for $b = a$. Recall that multiplication by p of Witt vectors coincides with $\text{Ver} \circ \sigma$. The left-hand side

$$\begin{aligned} \left(p^{\mu_{l+1}(a)} \sigma^{\mu_{l+1}(a)} D_a \right)^{(i, j + \mu_{l+1}(a))} &= \left(\text{Ver}^{\mu_{l+1}(a)} \sigma^{2\mu_{l+1}(a)} D_a \right)^{(i, j + \mu_{l+1}(a))} \\ &= \left(\sigma^{2\mu_{l+1}(a)} D_a \right)^{(ij)} = \left(D_a^{(ij)} \right)^{p^{2\mu_{l+1}(a)}} \end{aligned}$$

has degree $p^{2\mu_{l+1}(a)} \cdot p^{-\mu_{l+1}(a)+j} = p^{j+\mu_{l+1}(a)}$, and the right-hand side has strictly smaller degree by the previous paragraph. For all triples $(a, i, j) \in \Omega \setminus \Omega_l^\bullet$, interpret the $(i, j + \mu_{l+1}(a))$ -coordinate of Equation (3.3) as a rewriting rule (from left to right): any monomial in \mathcal{R} which is divisible by $\left(D_a^{(ij)} \right)^{p^{2\mu_{l+1}(a)}}$ can be rewritten by replacing this factor with the right-hand

side. All the rules obtained for all triples $(a, i, j) \in \Omega \setminus \Omega_l^\bullet$ define a rewriting system on \mathcal{R} . The strict inequality of degrees shown above together with the well-foundedness of the set of possible degrees of monomials implies that this rewriting system is strongly normalizing.

Consider an arbitrary $P \in \mathcal{R}$, and let $\tilde{P} \in \mathcal{R}$ be a normal form of P , obtained by rewriting P arbitrarily until it cannot be rewritten any further. We have $P \equiv \tilde{P} \pmod{\mathcal{I}_{l+1}}$ by definition of \mathcal{I}_{l+1} and of the rewriting system. By definition, \tilde{P} cannot be rewritten and thus does not contain any monomial divisible by $\left(D_a^{(ij)}\right)^{p^{2\mu_{l+1}(a)}}$ for any $(a, i, j) \in \Omega \setminus \Omega_l^\bullet$. This means that \tilde{P} belongs to the \mathcal{R}_l^\bullet -module spanned by monomials of the form given in Equation (4.5), concluding the proof. \square

Proposition 4.12 yields the following corollary, which implies Theorem 4.2 (c):

Corollary 4.13. *For every tuple $(d_{a,i,j})_{(a,i,j) \in \Omega_l^\bullet} \in \kappa^{|\Omega_l^\bullet|}$, we have*

$$\left| \left\{ D \in \mathfrak{g} \otimes \mathcal{D} \mid \begin{array}{l} D_a^{(ij)} = d_{a,i,j} \text{ for all } (a, i, j) \in \Omega_l^\bullet \\ \text{lastjump}(D) < l + 1 \end{array} \right\} \right| \leq |G|^{2l}.$$

In particular, we have Theorem 4.2 (c) with $E = |G|^2$:

$$|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < l + 1\}| \leq |G|^{2l} |\kappa|^{|\Omega_l^\bullet|} \leq |G|^{2l} |\kappa|^{rl}.$$

Proof. Denote by $f_l: \mathcal{R}_l^\bullet \rightarrow \mathcal{R}/\mathcal{I}_{l+1}$ the ring homomorphism obtained as the composition $\mathcal{R}_l^\bullet \hookrightarrow \mathcal{R} \rightarrow \mathcal{R}/\mathcal{I}_{l+1}$. By Proposition 4.12, the map f_l is finite of degree at most $|G|^{2l}$. The tuple $d_{a,i,j}$ corresponds to a ring homomorphism $d: \mathcal{R}_l^\bullet \rightarrow \kappa$ (induced by $D_a^{(ij)} \mapsto d_{a,i,j}$), and thus to a maximal ideal $\mathfrak{m} \subseteq \mathcal{R}_l^\bullet$ (namely $\mathfrak{m} := \ker(d)$) whose residue field is a subfield of κ (the image of d). We have:

$$\begin{aligned} & \left| \left\{ D \in \mathfrak{g} \otimes \mathcal{D} \mid \begin{array}{l} D_a^{(ij)} = d_{a,i,j} \text{ for all } (a, i, j) \in \Omega_l^\bullet \\ \text{lastjump}(D) < l + 1 \end{array} \right\} \right| \\ &= |\{D: \mathcal{R}/\mathcal{I}_{l+1} \rightarrow \kappa \mid D \circ f_l = d\}| && \text{by definition of } \mathcal{I}_{l+1} \\ &= \left| \left\{ \mathfrak{n} \text{ maximal ideal of } \mathcal{R}/\mathcal{I}_{l+1} \mid \begin{array}{l} f_l^{-1}(\mathfrak{n}) = \mathfrak{m} \\ (\mathcal{R}/\mathcal{I}_{l+1})/\mathfrak{n} \subseteq \kappa \end{array} \right\} \right| \\ &\leq \text{rank}_{\mathcal{R}_l^\bullet} \mathcal{R}/\mathcal{I}_{l+1} && \text{by Lemma 4.14 below} \\ &\leq |G|^{2l} && \text{by Proposition 4.12. } \quad \square \end{aligned}$$

In the proof, we have used the following lemma:

Lemma 4.14. *Let $f: A \rightarrow B$ be a finite ring homomorphism, and let \mathfrak{m} be a maximal ideal of A . Then, the number of maximal ideals of B containing $f(\mathfrak{m})$ is at most $\text{rank}_A B$.*

Proof. Replacing A by A/\mathfrak{m} and B by $B \otimes_A A/\mathfrak{m}$ (this may only decrease the rank of B), we may assume that A is a field and that $\mathfrak{m} = 0$. Since B is finite over the field A , it is Artinian and hence has finitely many maximal ideals. We want to show that B has at most $\dim_A B$ maximal ideals. The Chinese remainder theorem gives an isomorphism

$$\bigoplus_{\mathfrak{n} \subseteq B \text{ maximal}} B/\mathfrak{n} \simeq B/J(B)$$

where $J(B)$ is the Jacobson radical of B . We have

$$|\{\mathfrak{n} \subseteq B \text{ maximal}\}| \leq \sum_{\mathfrak{n} \subseteq B \text{ maximal}} \dim_A(B/\mathfrak{n}) = \dim_A(B/J(B)) \leq \dim_A B. \quad \square$$

Proving Theorem 4.2 (d). We now assume that $\mathfrak{g}[p]$ is non-abelian.

Proof of Theorem 4.2 (d)(i). Let $l \geq p$. For any $x, y \in \mathfrak{g}[p^2]$, we have $[px, py] = [p^2x, y] = 0$, so $p \cdot \mathfrak{g}[p^2]$ is an abelian subalgebra of $\mathfrak{g}[p]$. Since $\mathfrak{g}[p]$ is non-abelian, the subalgebra $p \cdot \mathfrak{g}[p^2]$ is strictly contained in $\mathfrak{g}[p]$, which implies that some exponent n_i in the decomposition $\mathfrak{g} \simeq \prod_i \mathbb{Z}/p^{n_i}\mathbb{Z}$

equals 1. Since $l \geq p$, we have $\mu_{l+1}(1) \geq 2$. Therefore, the inequality in Equation (4.4) is strict: we have $|\Omega_{l+1}^\bullet| \leq rl - 1$, and Corollary 4.13 gives the desired bound. \square

Proof of Theorem 4.2 (d)(ii). Let $l \in \{1, \dots, p-1\}$. We apply Corollary 4.3 with $v = l + lp^{-1}$. For $b \neq l$, Equations (4.1) and (4.2) are identical to the equations obtained for $v = l + 1$, which were analyzed above. For $b = l$, Equation (4.1) says that $p\sigma(D_l) = 0$ (the sum is empty because $a_1, a_2 < v$ implies that $a_1 + a_2 \leq 2l < lp$), so $D_l \in \mathfrak{g}[p] \otimes \kappa$. Equation (4.3) says that $[\sigma(D_l), D_l] = 0$ (only $i = 1$ is possible).

To restrict the possibilities for D_l , we look for an upper bound on the number of elements of $x \in \mathfrak{g}[p] \otimes \kappa$ satisfying $[\sigma(x), x] = 0$. For $x, y \in \mathfrak{g}[p] \otimes \kappa$, we can interpret $[x, y] = 0$ as a system of bilinear polynomial equations in the r coordinates $x_i, y_i \in \kappa$ of x and y . By assumption, the Lie \mathbb{F}_p -algebra $\mathfrak{g}[p]$ is non-abelian, so one of these polynomial equations is non-trivial. Consider any non-zero monomial $cx_i y_j$ in this equation. Substituting $x = \sigma(y)$, the resulting polynomial equation (in the r variables y_i) has a non-zero monomial $cy_i^p y_j$, so in particular the closed subvariety (of r -dimensional affine space) cut out by the equation $[\sigma(y), y]$ has dimension at most $r - 1$. By [LW54, Lemma 1], the number of solutions $D_l \in \mathfrak{g}[p] \otimes \kappa$ to $[\sigma(D_l), D_l] = 0$ is then $\mathcal{O}(|\kappa|^{r-1})$ (where the implied constant depends only on $\mathfrak{g}[p]$).

Consider one such solution $D_l \in \mathfrak{g}[p] \otimes \kappa$. For the other elements $D_b \in \mathfrak{g} \otimes W(\kappa)$ (with $b \neq l$), we reason exactly like in the proof of Theorem 4.2 (c), using Corollary 4.13. The analogue of Ω_l^\bullet is now given by the set $\{(a, i, j) \in \Omega \mid a \neq l \text{ and } n_i - \mu_{l+1}(a) \leq j\}$ which has cardinality

$$\sum_{\substack{a \neq l \\ a \in \mathbb{N} \setminus p\mathbb{N}}} \sum_{i=1}^r \min(n_i, \mu_{l+1}(a)) \leq r \sum_{\substack{a \neq l \\ a \in \mathbb{N} \setminus p\mathbb{N}}} \mu_{l+1}(a) = r \left(l - \underbrace{\mu_{l+1}(l)}_{=1} \right) = r(l-1).$$

Finally, we obtain the desired bound on the total number of solutions:

$$\begin{aligned} |\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < l(1 + p^{-1})\}| &= \mathcal{O} \left(\underbrace{|\kappa|^{r-1}}_{\text{choices of } D_l} \cdot \underbrace{E^{l+lp^{-1}-1} |\kappa|^{r(l-1)}}_{\text{choices of } (D_b)_{b \neq l} \text{ once } D_l \text{ is fixed}} \right) \\ &= \mathcal{O} \left(E^l |\kappa|^{rl-1} \right). \end{aligned} \quad \square$$

Refinement of the value of E . Finally, we note that the bound of Theorem 4.2 (points (c) and (d)) can sometimes² be refined:

Proposition 4.15. *Assume that there is a Lie subalgebra $\mathfrak{n} \subseteq Z(\mathfrak{g})$ containing $[\mathfrak{g}, \mathfrak{g}]$ such that*

$$\mathfrak{n} \cap p^n \mathfrak{g} = p^n \mathfrak{n} \quad \text{for any } n \geq 1. \quad (4.6)$$

Then, for all integers $l \geq 0$, we have Theorem 4.2 (c) with $E = 1$:

$$|\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < l + 1\}| \leq |\kappa|^{rl}.$$

Equation (4.6) is automatically satisfied for $n \geq \max_i n_i$. In particular, the proposition applies automatically (for example, taking $\mathfrak{n} = Z(\mathfrak{g})$) when $\mathfrak{g} = \mathfrak{g}[p]$ is of exponent p . (This case will be discussed again in Subsection 6.2.) It also applies if \mathfrak{g} is abelian, taking $\mathfrak{n} = \mathfrak{g}$.

Proof. Modulo $\mathfrak{n} \otimes W(\kappa)$, Equation (3.3) becomes $p^{\mu_{l+1}(b)} \bar{D}_b = 0$ for all b (as \mathfrak{n} contains $[\mathfrak{g}, \mathfrak{g}]$). First choose $\bar{D}_b \in (\mathfrak{g}/\mathfrak{n})[p^{\mu_{l+1}(b)}] \otimes W(\kappa)$ for all b . An upper bound for the number of choices for \bar{D}_b is

$$\prod_{b \in \mathbb{N} \setminus p\mathbb{N}} \left| (\mathfrak{g}/\mathfrak{n})[p^{\mu_{l+1}(b)}] \otimes W(\kappa) \right|.$$

Having determined \bar{D}_b modulo $\mathfrak{n} \otimes W(\kappa)$ for all b (and hence modulo $Z(\mathfrak{g}) \otimes W(\kappa)$), it follows from Equation (3.3) that $p^{\mu_{l+1}(b)} D_b$ is determined for all b . Hence, D_b is determined modulo

²We did not manage to find any Lie algebra for which the inequality does not hold with $E = 1$.

$(\mathfrak{g}[p^{\mu_{l+1}(b)}] \cap \mathfrak{n}) \otimes W(\kappa) = \mathfrak{n}[p^{\mu_{l+1}(b)}] \otimes W(\kappa)$. An upper bound for the number of remaining choices for D_b (once \bar{D}_b is fixed for all b) is

$$\prod_{b \in \mathbb{N} \setminus p\mathbb{N}} \left| \mathfrak{n}[p^{\mu_{l+1}(b)}] \otimes W(\kappa) \right|.$$

Note the short exact sequence (for any $n \geq 0$)

$$0 \rightarrow \mathfrak{g}[p^n]/\mathfrak{n}[p^n] \rightarrow (\mathfrak{g}/\mathfrak{n})[p^n] \xrightarrow{\times p^n} (\mathfrak{n} \cap p^n \mathfrak{g})/p^n \mathfrak{n} \rightarrow 0.$$

By hypothesis, the quotient $(\mathfrak{n} \cap p^n \mathfrak{g})/p^n \mathfrak{n}$ is trivial, so that $(\mathfrak{g}/\mathfrak{n})[p^n] = \mathfrak{g}[p^n]/\mathfrak{n}[p^n]$. Therefore, the total number of choices for D_b is bounded above by

$$\begin{aligned} \prod_{b \in \mathbb{N} \setminus p\mathbb{N}} \left| (\mathfrak{g}[p^{\mu_{l+1}(b)}]/\mathfrak{n}[p^{\mu_{l+1}(b)}]) \otimes W(\kappa) \right| \cdot \left| \mathfrak{n}[p^{\mu_{l+1}(b)}] \otimes W(\kappa) \right| &= \prod_{b \in \mathbb{N} \setminus p\mathbb{N}} \left| \mathfrak{g}[p^{\mu_{l+1}(b)}] \otimes W(\kappa) \right| \\ &= |\kappa|^{\sum \log_p |\mathfrak{g}[p^{\mu_{l+1}(b)}]|}. \end{aligned}$$

It remains only to estimate the exponent, using the inequalities $|\mathfrak{g}[p^k]| \leq |\mathfrak{g}[p]|^k$ for $k \geq 0$:

$$\sum_{b \in \mathbb{N} \setminus p\mathbb{N}} \log_p |\mathfrak{g}[p^{\mu_{l+1}(b)}]| \leq \sum_{b \in \mathbb{N} \setminus p\mathbb{N}} r \mu_{l+1}(b) \stackrel{\text{Lemma 3.8}}{=} r l. \quad \square$$

5. GLOBAL ASYMPTOTICS

We fix a rational function field $F := \mathbb{F}_q(T)$ of characteristic p and a finite Lie \mathbb{Z}_p -algebra $\mathfrak{g} \neq 0$ of nilpotency class at most 2. Let $G := (\mathfrak{g}, \circ)$ be the corresponding p -group. For each place P of $F = \mathbb{F}_q(T)$, we make the following definitions:

- F_P is the completion of F at P , κ_P is its residue field, and π_P is a uniformizer of F_P , so that $F_P = \kappa_P((\pi_P))$. Note that $|\kappa_P| = q^{\deg(P)}$ is the absolute norm of the prime P ;
- $\tilde{\pi}_P = (\pi_P, 0, 0, \dots)$ is the Teichmüller representative of π_P in $W(F_P) \subseteq W(F_P^{\text{perf}})$.
- The following objects are defined exactly as in Section 3 in the case $(\mathfrak{F}, \pi) = (F_P, \pi_P)$, adding a subscript P to the notation:
 - The $W(\kappa_P)$ -submodules \mathcal{D}_P^0 and \mathcal{D}_P of $W(F_P) \subseteq W(F_P^{\text{perf}})$ (cf. Definitions 3.2 and 3.14)
 - The maps $\alpha_P^0: \mathfrak{g} \otimes W(F_P^{\text{perf}}) //_{W(F_P^{\text{perf}})} \xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{D}_P^0 //_{W(\kappa_P)}$ (cf. Theorem 3.6) and $\alpha_P: \mathfrak{g} \otimes W(F_P^{\text{perf}}) //_{W(F_P^{\text{perf}})} \rightarrow \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$ (cf. Definition 3.18).

We also use the notation $\text{lastjump}(D)$ if $D \in \mathfrak{g} \otimes \mathcal{D}_P^0$ (cf. Definition 3.7) or if $D \in \mathfrak{g} \otimes \mathcal{D}_P$ (cf. Definition 3.24).

This section is organized as follows: In Subsection 5.1, we prove the local–global principle (Theorem 1.1). Then, in Subsection 5.2, we establish a general analytic lemma (Lemma 5.4), which lets one deduce global asymptotics from local estimates when combined with the local–global principle. Finally, we prove our main counting result (Theorem 1.2) in Subsection 5.3. Note that Theorem 1.1 and Lemma 5.4 are also used in Section 6 to prove Theorem 6.7.

5.1. Local–global principle. Before proving Theorem 1.1, we introduce some notation. At each place P , the inclusion $F \hookrightarrow F_P$ induces a map $\mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \rightarrow \mathfrak{g} \otimes W(F_P^{\text{perf}}) //_{W(F_P^{\text{perf}})}$, corresponding to the natural map $H^1(\Gamma_F, G) \rightarrow H^1(\Gamma_{F_P}, G)$ (see Remark 2.22 (b)). Combining this map with the maps $\alpha_P: \mathfrak{g} \otimes W(F_P^{\text{perf}}) //_{W(F_P^{\text{perf}})} \rightarrow \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$ for all places P , we define a map $\mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \rightarrow \prod_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$.

Consider an element $[m] \in \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$, and let K be the associated G -extension of F . For all but finitely many places P of F , the extension $K|F$ is unramified, so the corresponding element of $\mathfrak{g} \otimes W(F_P^{\text{perf}}) //_{W(F_P^{\text{perf}})}$ (the $(\mathfrak{g} \otimes W(F_P^{\text{perf}}), \circ)$ -orbit of m) is mapped by

$\alpha_P: \mathfrak{g} \otimes W(F_P^{\text{perf}}) //_{W(F_P^{\text{perf}})} \rightarrow \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$ to the trivial orbit $\{0\}$ (see Lemma 4.5). We have thus described a map

$$\alpha: \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \longrightarrow \prod'_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}, \quad (5.1)$$

where the restricted product $\prod'_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$ is the following subset of $\prod_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$:

$$\prod'_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)} := \left\{ ([D_P])_P \text{ place of } F \mid [D_P] = \{0\} \text{ for all but finitely many places } P \right\}.$$

Lemma 5.1. *For every $([D_P]) \in \prod'_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$, we have*

$$\sum_{\substack{[\gamma] \in H^1(\Gamma_F, G): \\ \alpha(\text{orb}([\gamma])) = ([D_P])}} \frac{1}{|\text{Stab}_G(\gamma)|} = \prod_P |[D_P]|, \quad (5.2)$$

where $|[D_P]|$ denotes the size of the $(\mathfrak{g} \otimes W(\kappa_P), \circ)$ -orbit of D_P .

Proof. We first prove the claim in the case $\mathfrak{g} = \mathbb{Z}/p\mathbb{Z}$, where $G = \mathbb{Z}/p\mathbb{Z}$. In this case, the orbits $[D_P]$ all have size 1 by Proposition 3.16 (ii). Let $\mathcal{O}_P = \kappa_P[[\pi_P]]$ be the ring of integers of F_P . We have the following exact sequence, established in [Pot26, Corollary 5.3]:

$$0 \longrightarrow \mathbb{F}_q / \wp(\mathbb{F}_q) \longrightarrow F / \wp(F) \xrightarrow{\alpha} \bigoplus_P F_P / (\mathcal{O}_P + \wp(F_P)) \longrightarrow 0. \quad (5.3)$$

Here, we have identified the surjection $F / \wp(F) \rightarrow \bigoplus_P F_P / (\mathcal{O}_P + \wp(F_P))$ with the map α of Equation (5.1) as follows:

- On the one hand, we have an identification

$$F / \wp(F) \underset{\text{Lemma 2.9}}{\simeq} F^{\text{perf}} / \wp(F^{\text{perf}}) = \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}.$$

- On the other hand, we have identifications

$$F_P / \wp(F_P) \underset{\text{Lemma 2.9}}{\simeq} F_P^{\text{perf}} / \wp(F_P^{\text{perf}}) = \mathfrak{g} \otimes W(F_P^{\text{perf}}) //_{W(F_P^{\text{perf}})} \underset{\text{Theorem 3.6}}{\simeq} \mathfrak{g} \otimes \mathcal{D}_P^0 //_{W(\kappa_P)}.$$

Recall that $\mathfrak{g} \otimes \mathcal{D}_P^0 = \mathcal{D}_P^0 / p\mathcal{D}_P^0 = \bigoplus_{a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}} \kappa_P \pi_P^{-a}$, so $\mathfrak{g} \otimes \mathcal{D}_P^0 //_{W(\kappa_P)} = (\mathcal{D}_P^0 / p\mathcal{D}_P^0) / \wp(\kappa_P)$. Similarly, $\mathfrak{g} \otimes \mathcal{D}_P = \bigoplus_{a \in \mathbb{N} \setminus p\mathbb{N}} \kappa_P \pi_P^{-a} = \mathfrak{g} \otimes \mathcal{D}_P^0 / \kappa_P$. As $\mathcal{O}_P \cap (\mathcal{D}_P^0 / p\mathcal{D}_P^0) = \kappa_P$, we obtain an identification

$$F_P / (\mathcal{O}_P + \wp(F_P)) \simeq \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}.$$

Assembling the identifications $F_P / (\mathcal{O}_P + \wp(F_P)) \simeq \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$ for all places P yields the desired identification $\bigoplus_P F_P / (\mathcal{O}_P + \wp(F_P)) \simeq \prod'_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$.

Using the exact sequence (5.3), one sees that there are exactly $|\mathbb{F}_q / \wp(\mathbb{F}_q)| = p$ cosets $[m] \in F / \wp(F)$ with $\alpha([m]) = ([D_P])$. Since $\text{orb} : H^1(\Gamma_F, G) \rightarrow F / \wp(F)$ is a bijection, there are exactly p classes $[\gamma] \in H^1(\Gamma_F, G)$ with $\alpha(\text{orb}([\gamma])) = ([D_P])$ for all places P . Since G is abelian, each of them has stabilizer $\text{Stab}_G(\gamma) = G$ of size p . The claim then follows:

$$\sum_{\substack{[\gamma] \in H^1(\Gamma_F, G): \\ \alpha(\text{orb}([\gamma])) = ([D_P])}} \frac{1}{|\text{Stab}_G(\gamma)|} = p \cdot \frac{1}{p} = 1 = \prod_P |[D_P]|.$$

We now treat the case of a general Lie algebra \mathfrak{g} . Let A be the left-hand side of Equation (5.2). By the orbit-stabilizer theorem, we have

$$A = \frac{1}{|G|} \left| \left\{ \gamma \in \text{Hom}(\Gamma_F, G) \mid \alpha(\text{orb}([\gamma])) = ([D_P]) \right\} \right|.$$

We show that $A = \prod_P |[D_P]|$ by induction over the size of the non-zero finite Lie algebra \mathfrak{g} . Pick any Lie subalgebra $\mathfrak{n} \subseteq Z(\mathfrak{g})$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$, and let $N := (\mathfrak{n}, \circ)$ be the corresponding subgroup of $Z(G)$. We denote the corresponding projection maps (modulo $\mathfrak{n} \otimes \mathcal{D}_P$ or N) by π . Since \mathcal{D}_P is a flat \mathbb{Z}_p -module, we have an exact sequence

$$0 \rightarrow \mathfrak{n} \otimes \mathcal{D}_P \rightarrow \mathfrak{g} \otimes \mathcal{D}_P \xrightarrow{\pi} (\mathfrak{g}/\mathfrak{n}) \otimes \mathcal{D}_P \rightarrow 0.$$

We can write

$$A = \frac{1}{|G/N|} \sum_{\substack{\varepsilon \in \text{Hom}(\Gamma_F, G/N): \\ \alpha(\text{orb}([\varepsilon])) = ([\pi(D_P)])}} a(\varepsilon)$$

where we have defined

$$a(\varepsilon) := \frac{1}{|N|} \left| \left\{ \gamma \in \text{Hom}(\Gamma_F, G) \mid \pi \circ \gamma = \varepsilon \text{ and } \alpha(\text{orb}([\gamma])) = ([D_P]) \right\} \right|.$$

Let $\varepsilon \in \text{Hom}(\Gamma_F, G/N)$ be such that $\alpha(\text{orb}([\varepsilon])) = ([\pi(D_P)])$. The surjectivity of the natural map $\mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \rightarrow (\mathfrak{g}/\mathfrak{n}) \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$ implies the surjectivity of the corresponding map $H^1(\Gamma_F, G) \rightarrow H^1(\Gamma_F, G/N)$ (cf. Proposition 2.21 (a)), and therefore of $\text{Hom}(\Gamma_F, G) \rightarrow \text{Hom}(\Gamma_F, G/N)$. (This reflects the fact that there is no obstruction to the embedding problem for $G \twoheadrightarrow G/N$.) Thus, we can pick an arbitrary preimage $\gamma_0 \in \text{Hom}(\Gamma_F, G)$ such that $\pi \circ \gamma_0 = \varepsilon$. The entire fiber $\{\gamma \in \text{Hom}(\Gamma_F, G) \mid \pi \circ \gamma = \varepsilon\}$ can then be described as the set of twists $\gamma_0 \cdot \delta$ with $\delta \in \text{Hom}(\Gamma_F, N)$ (cf. Remark 2.3). We therefore have

$$a(\varepsilon) = \frac{1}{|N|} \left| \left\{ \delta \in \text{Hom}(\Gamma_F, N) \mid \alpha(\text{orb}([\gamma_0 \cdot \delta])) = ([D_P]) \right\} \right|.$$

Since $\alpha(\text{orb}([\varepsilon])) = ([\pi(D_P)])$ and γ_0 lifts ε , we have $\alpha(\text{orb}([\gamma_0])) = ([U_P]) \in \prod'_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$ for some $U_P \in \mathfrak{g} \otimes \mathcal{D}_P$ with $\pi(U_P) = \pi(D_P)$. If, for a given $\delta \in \text{Hom}(\Gamma_F, N)$, we write $\alpha(\text{orb}([\delta])) = ([V_P]) \in \prod'_P \mathfrak{n} \otimes \mathcal{D}_P //_{W(\kappa_P)}$, then $\alpha(\text{orb}([\gamma_0 \cdot \delta])) = ([U_P + V_P]) \in \prod'_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$ by Lemma 2.24 and Lemma 3.19. We obtain

$$\begin{aligned} a(\varepsilon) &= \sum_{\substack{([V_P]) \in \prod'_P \mathfrak{n} \otimes \mathcal{D}_P //_{W(\kappa_P)}: \\ ([U_P + V_P]) = ([D_P])}} \underbrace{\frac{1}{|N|} \left| \left\{ \delta \in \text{Hom}(\Gamma_F, N) \mid \alpha(\text{orb}([\delta])) = ([V_P]) \right\} \right|}_{=1 \text{ by the base case, as } \mathfrak{n} \simeq \mathbb{Z}/p\mathbb{Z}} \\ &= \left| \left\{ ([V_P]) \in \prod'_P \mathfrak{n} \otimes \mathcal{D}_P //_{W(\kappa_P)} \mid ([U_P + V_P]) = ([D_P]) \right\} \right| \\ &= \prod'_P \left| \left\{ V_P \in \mathfrak{n} \otimes \mathcal{D}_P \mid U_P + V_P \in [D_P] \right\} \right| && \text{by Proposition 3.16 (ii)} \\ &= \prod'_P |\{x \in [D_P] \mid x - U_P \in \mathfrak{n} \otimes \mathcal{D}_P\}| && \text{by the change of variables } x = U_P + V_P \\ &= \prod'_P |\{x \in [D_P] \mid \pi(x) = \pi(U_P)\}| \end{aligned}$$

The number of such x is the size of the fiber of the map $\pi: [D_P] \rightarrow [\pi(D_P)]$, $x \mapsto \pi(x)$ above $\pi(U_P) = \pi(D_P)$. All the fibers of that map have the same size since $[D_P]$ is a single $(\mathfrak{g} \otimes W(\kappa_P), \circ)$ -orbit. Hence

$$a(\varepsilon) = \prod'_P \frac{|[D_P]|}{|[\pi(D_P)]|}.$$

In particular, $a(\varepsilon)$ does not depend on ε . We can finally conclude:

$$A = \frac{1}{|G/N|} \underbrace{\left| \left\{ \varepsilon \in \text{Hom}(\Gamma_F, G/N) \mid \alpha(\text{orb}([\varepsilon])) = ([\pi(D_P)]) \right\} \right|}_{=\prod'_P |[\pi(D_P)]| \text{ by the induction hypothesis}} \cdot \prod'_P \frac{|[D_P]|}{|[\pi(D_P)]|} = \prod'_P |[D_P]|. \quad \square$$

The local–global principle for the last jump (Theorem 1.1) follows readily from Lemma 5.1:

Theorem 5.2 (Local–global principle). *Pick rational numbers $N_P \in \mathbb{Q}_{\geq 0}$ for every place P of $F = \mathbb{F}_q(T)$, so that $N_P = 0$ for all but finitely many places P . Then,*

$$\sum_{\substack{K \in \text{ÉtExt}(G, F): \\ \forall P, \text{lastjump}_P(K) = N_P}} \frac{1}{|\text{Aut}(K)|} = \prod'_P \sum_{\substack{K_P \in \text{ÉtExt}(G, F_P): \\ \text{lastjump}(K_P) = N_P}} \frac{1}{|\text{Aut}(K_P)|}.$$

Proof. Using the bijection $\acute{\text{E}}\text{tExt}(G, F) \xrightarrow{\sim} H^1(\Gamma_F, G)$ from Lemma 2.1 and the fact that $\text{lastjump}_P(K) = \text{lastjump}(D_P)$ if K corresponds to $[\gamma]$ and $\alpha_P(\text{orb}([\gamma])) = [D_P]$ (cf. Definitions 2.23, 3.7 and 3.24), we have

$$\sum_{\substack{K \in \acute{\text{E}}\text{tExt}(G, F): \\ \forall P, \text{lastjump}_P(K) = N_P}} \frac{1}{|\text{Aut}(K)|} = \sum_{\substack{([D_P]) \in \prod'_P \mathfrak{g} \otimes \mathcal{D}_P // W(\kappa_P): \\ \forall P, \text{lastjump}([D_P]) = N_P}} \sum_{\substack{[\gamma] \in H^1(\Gamma_F, G): \\ \alpha(\text{orb}([\gamma])) = ([D_P])}} \frac{1}{|\text{Stab}_G(\gamma)|}$$

By Lemma 5.1, the inner sum equals $\prod_P |[D_P]|$. Therefore:

$$\begin{aligned} \sum_{\substack{K \in \acute{\text{E}}\text{tExt}(G, F): \\ \forall P: \text{lastjump}_P(K) = N_P}} \frac{1}{|\text{Aut}(K)|} &= \prod_P |\{D_P \in \mathfrak{g} \otimes \mathcal{D}_P \mid \text{lastjump}(D_P) = N_P\}| \\ &= \prod_P \sum_{\substack{K_P \in \acute{\text{E}}\text{tExt}(G, F_P): \\ \text{lastjump}(K_P) = N_P}} \frac{1}{|\text{Aut}(K_P)|} \quad \text{by Lemma 4.1. } \square \end{aligned}$$

Remark 5.3. In the proof of the local–global principle, we crucially made use of the fact that, for any $D \in \mathfrak{g} \otimes \mathcal{D}^0$, the number $\text{lastjump}(D)$ is completely determined by the projection $\text{pr}(D) \in \mathfrak{g} \otimes \mathcal{D}$. (See Corollary 3.23.) Our method of proof does not imply a similar local–global principle for the discriminant, as the discriminant in general cannot be determined from $\text{pr}(D)$. For the same reason, our method of proof fails for groups of nilpotency class larger than 2.

5.2. Analytic lemma. We now prove a general analytic lemma that allows us to combine local estimates into global asymptotics when there is a local–global principle. We write the lemma and its proof in such a way that it is valid for any function field F of characteristic p with field of constants \mathbb{F}_q , without having to assume that $F = \mathbb{F}_q(T)$.

Lemma 5.4. *Let $K \geq 1$ be an integer. For every place P of F and every $n \in \frac{1}{K}\mathbb{Z}_{\geq 0}$, let $a_{P,n} \in \mathbb{R}_{\geq 0}$, with $a_{P,0} = 1$. For every $n \in \frac{1}{K}\mathbb{Z}_{>0}$, let $b_n \in \mathbb{Z}_{\geq 0}$ and $e_n, k_n \in \mathbb{R}$ be such that for all places P , we have*

$$a_{P,n} = b_n |\kappa_P|^{e_n} + \mathcal{O}(|\kappa_P|^{k_n}),$$

where the implied constant factor in the \mathcal{O} -estimate depends neither on the place P nor on the number n . Assume that $b_n = 0$ for all but finitely many numbers n , but $b_n \neq 0$ for at least one number n . Now, define

$$A := \max \left\{ \frac{e_n + 1}{n} \mid n \in \frac{1}{K}\mathbb{Z}_{>0} \text{ with } b_n \neq 0 \right\}$$

and assume that

$$\sup \left\{ \frac{k_n + 1}{n} \mid n \in \frac{1}{K}\mathbb{Z}_{>0} \right\} < A. \quad (5.4)$$

Let

$$S := \{n \in \frac{1}{K}\mathbb{Z}_{>0} \mid b_n \neq 0 \text{ and } A = \frac{e_n + 1}{n}\},$$

define

$$B := \sum_{n \in S} b_n, \quad (5.5)$$

and let $M \in \frac{1}{K}\mathbb{Z}_{>0}$ be the least common integer multiple of the numbers $n \in S$, so that $M\mathbb{Z} = \bigcap_{n \in S} n\mathbb{Z}$. Finally, define

$$a_N := \sum_{\substack{(n_P)_P \in \prod_P \frac{1}{K}\mathbb{Z}_{\geq 0}: \\ \sum_P n_P \deg(P) = N}} \prod_P a_{P, n_P}.$$

Then, there is a function $C : \mathbb{Q}/M\mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ with $C(0) > 0$, such that for rational N going to infinity, we have

$$a_N = C(N \bmod M) \cdot q^{AN} N^{B-1} + o(q^{AN} N^{B-1}).$$

Proof. Rescaling by a factor of K , we can assume without loss of generality that $K = 1$. We then only need to consider integers N since $a_N = 0$ for $N \notin \mathbb{Z}_{\geq 0}$. Moreover, we can assume without loss of generality that $b_n = 0$ for all $n \notin S$. (For the finitely many $n \notin S$ with $b_n \neq 0$, replace b_n by 0 and k_n by $\max(e_n, k_n)$, and observe that Inequality (5.4) still holds.) Let $\delta := A - \sup\{\frac{k_n+1}{n} \mid n \in \mathbb{Z}_{>0}\} > 0$. By definition of the numbers a_N , the generating function $f(X) := \sum_{N \geq 0} a_N X^N$ factors as an Euler product

$$f(X) = \prod_P f_P(X^{\deg(P)}), \quad \text{where} \quad f_P(X) := \sum_{n \geq 0} a_{P,n} X^n.$$

We estimate the local factor at a place P , for a complex number X :

$$\begin{aligned} f_P(|\kappa_P|^{-A} X) &= 1 + \sum_{n \geq 1} a_{P,n} |\kappa_P|^{-An} X^n \\ &= 1 + \sum_{n \in S} b_n |\kappa_P|^{e_n - An} X^n + \sum_{n \geq 1} \mathcal{O}(|\kappa_P|^{k_n - An} X^n) \\ &= 1 + \sum_{n \in S} b_n |\kappa_P|^{-1} X^n + \sum_{n \geq 1} \mathcal{O}(|\kappa_P|^{-1 - \delta n} X^n). \end{aligned}$$

Recall that the constant in the \mathcal{O} -estimate is independent of both n and P . We obtain

$$\begin{aligned} f_P(|\kappa_P|^{-A} X) &= 1 + \sum_{n \in S} b_n |\kappa_P|^{-1} X^n + \mathcal{O}\left(\sum_{n \geq 1} |\kappa_P|^{-1 - \delta n} X^n\right) \\ &= 1 + |\kappa_P|^{-1} \sum_{n \in S} b_n X^n + |\kappa_P|^{-1 - \delta} X \cdot \mathcal{O}\left(\sum_{n \geq 0} (|\kappa_P|^{-\delta} X)^n\right) \end{aligned}$$

For $|X| < |\kappa_P|^{\delta/2}$, we have $|\kappa_P|^{-\delta} |X| \leq |\kappa_P|^{-\delta/2} \leq q^{-\delta/2}$, and then

$$\left| \sum_{n \geq 0} (|\kappa_P|^{-\delta} X)^n \right| \leq \sum_{n \geq 0} q^{-n\delta/2} = \frac{1}{1 - q^{-\delta/2}},$$

which means that the \mathcal{O} -factor is bounded by a constant independent of P . Note also that $|\kappa_P|^{-1 - \delta} |X| \leq |\kappa_P|^{-1 - \delta/2}$. We have obtained the estimate

$$f_P(|\kappa_P|^{-A} X) = 1 + |\kappa_P|^{-1} \sum_{n \in S} b_n X^n + \mathcal{O}(|\kappa_P|^{-1 - \delta/2}),$$

which implies, under the additional assumption that $|X| < |\kappa_P|^{1/3n}$ for all $n \in S$, that

$$f_P(|\kappa_P|^{-A} X) \prod_{n \in S} (1 - |\kappa_P|^{-1} X^n)^{b_n} = 1 + \mathcal{O}(|\kappa_P|^{-1 - \varepsilon})$$

for $\varepsilon := \min(\frac{1}{3}, \frac{\delta}{2})$. Let $\delta' := \min\left(\{\frac{\delta}{2}\} \cup \{\frac{1}{3n} \mid n \in S\}\right)$. For $|X| < q^{\delta'}$, we obtain

$$f(q^{-A} X) = \prod_P f_P\left(|\kappa_P|^{-A} X^{\deg(P)}\right) = \prod_P \frac{1 + \mathcal{O}(|\kappa_P|^{-1 - \varepsilon})}{\prod_{n \in S} (1 - (q^{-1} X^n)^{\deg(P)})^{b_n}}. \quad (5.6)$$

The product of the numerators is absolutely convergent and hence defines a holomorphic function for $|X| < q^{\delta'}$. Moreover, that product does not vanish for $X = 1$ as none of the factors $f_P(|\kappa_P|^{-A}) = 1 + \sum_{n \geq 1} a_{P,n} |\kappa_P|^{-An} \geq 1$ vanish (the coefficients $a_{P,n}$ are non-negative). The Hasse–Weil zeta function

$$Z_F(X) = \prod_P \frac{1}{1 - X^{\deg(P)}}$$

is holomorphic for $|X| \leq \frac{1}{q}$ except for a simple pole at $X = \frac{1}{q}$. Using this function, Equation (5.6), rewrites as

$$f(q^{-A}X) = \underbrace{\prod_P \left(1 + \mathcal{O}(|\kappa_P|^{-1-\varepsilon})\right)}_{\substack{\text{holomorphic for } |X| < q^{\delta'} \\ \text{non-vanishing at } X = 1}} \cdot \prod_{n \in S} \underbrace{Z_F(q^{-1}X^n)^{b_n}}_{\substack{\text{holomorphic for } |X| \leq 1, \text{ except for} \\ \text{poles of order } b_n \text{ at } n\text{-th roots of } 1}}$$

Thus, the function $f(q^{-A}X) \prod_{n \in S} (1 - X^n)^{b_n}$ is holomorphic for $|X| \leq 1$ and non-vanishing at $X = 1$. It follows that $f(X)$ is holomorphic for $|X| \leq q^{-A}$ except for poles of order at most B at $q^{-A}, q^{-A} \cdot \zeta, \dots, q^{-A} \cdot \zeta^{M-1}$ for the M -th roots of unity $1, \zeta, \dots, \zeta^{M-1}$, and the pole at q^{-A} has order exactly B . By [FS09, Theorem IV.10], this implies that

$$a_N = \sum_{i=0}^{M-1} D_i (q^{-A} \zeta^i)^{-N} N^{B-1} + o(q^{AN} N^{B-1})$$

for some constants D_0, \dots, D_{M-1} , with $D_0 \neq 0$. We conclude that

$$a_N = C(N) q^{AN} N^{B-1} + o(q^{AN} N^{B-1}),$$

where $C(N) := \sum_{i=0}^{M-1} D_i \zeta^{-iN}$ only depends on the remainder of N modulo M . Since $D_0 \neq 0$, we have $C(N) \neq 0$ for some N . Since $a_N \geq 0$ for all N , we have $C(N) \geq 0$ for all N .

We show $C(0) > 0$ as follows: Let ν be the greatest common divisor of the numbers $n \in S$. Let $L \geq 0$ be large enough so that every residue class modulo M which is divisible by ν can be written as the sum of at most L (not necessarily distinct) elements of S .

Inequality (5.4) implies $e_n > k_n$ for $n \in S$, so $a_{P,n} > 0$ for all but finitely many places P . By the Weil bound, for any sufficiently large integer d , there is a place of degree d . In particular, there are infinitely many places with $\deg(P) \equiv 1 \pmod{M}$. Pick any L places Q_1, \dots, Q_L with $\deg(Q_i) \equiv 1 \pmod{M}$ and $a_{Q_i,n} > 0$ for all $n \in S$, and let Ω be the set of remaining places $P \neq Q_1, \dots, Q_L$. Restricting to places in Ω and elements n_P of S , define

$$a'_N := \sum_{\substack{(n_P)_{P \in \prod_{P \in \Omega} S}: \\ \sum_{P \in \Omega} n_P \deg(P) = N}} \prod_{P \in \Omega} a_{P,n_P}.$$

We have $a'_N = C'(N \bmod M) \cdot q^{AN} N^{B-1} + o(q^{AN} N^{B-1})$ by the same argument as above, again with $C'(r) > 0$ for some residue class r modulo M . By definition, $a'_N = 0$ unless N is a sum of elements of S , and hence divisible by ν . Thus, $C'(r) > 0$ implies that r is divisible by ν . Now, write $-r \equiv n_{Q_1} + \dots + n_{Q_L} \pmod{M}$ with $n_{Q_1}, \dots, n_{Q_L} \in \{0\} \cup S$. The inequality

$$a_{n_{Q_1} \deg(Q_1) + \dots + n_{Q_L} \deg(Q_L) + N} \geq a_{Q_1, n_1} \cdots a_{Q_L, n_L} a'_N$$

for $N \equiv r \pmod{M}$ implies that $C(0) \geq a_{Q_1, n_1} \cdots a_{Q_L, n_L} C'(r) > 0$. \square

Remark 5.5. Note the similarities between our Equations (5.4) and (5.5) and [DY25, Equations (8.1) and (8.2)]. In that regard, our local-global principle (Theorem 5.2) seems to be compatible with [DY25, Conjecture 6.10].

5.3. Main counting result.

Theorem 5.6. *Let*

$$r := \log_p |G[p]| \quad \text{and} \quad M := \begin{cases} 1 & \text{if } G[p] \text{ is abelian,} \\ 1 + p^{-1} & \text{otherwise.} \end{cases}$$

If $G[p]$ is non-abelian, assume that $|G[p]| \leq p^{p-1}$. Assume moreover that q is a large enough power of p (depending on the group G). Then, there is a function $C : \mathbb{Q}/M\mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ with $C(0) \neq 0$, such that for rational $N \rightarrow \infty$, we have

$$\sum_{\substack{K \in \text{ÉtExt}(G, F): \\ \text{lastjump}(K) = N}} \frac{1}{|\text{Aut}(K)|} = C(N \bmod M) \cdot q^{\frac{r+1}{M} \cdot N} + o\left(q^{\frac{r+1}{M} \cdot N}\right).$$

Proof. For every $N \in \mathbb{Q}_{\geq 0}$, define

$$a_N := \sum_{\substack{K \in \text{ÉtExt}(G, F); \\ \text{lastjump}(K) = N}} \frac{1}{|\text{Aut}(K)|}.$$

Combining Equation (1.1), Theorem 5.2, and Lemma 4.1, we see that

$$a_N = \sum_{\substack{(n_P)_P \in \prod_P \frac{1}{|\mathfrak{g}|} \mathbb{Z}_{\geq 0}: \\ \sum_P n_P \deg(P) = N}} \prod_P a_{P, n_P} \quad \text{where} \quad a_{P, n} := |\{D \in \mathfrak{g} \otimes \mathcal{D}_P \mid \text{lastjump}(D) = n\}|.$$

Our goal is now to use Lemma 5.4 (with $K = |\mathfrak{g}|$) in order to obtain estimates for a_N . For this, we use the following estimates arising from Theorem 4.2:

If $\mathfrak{g}[p]$ is non-abelian, then:

$$\begin{aligned} a_{P, n} &= 1 && \text{for } n = 0, \\ a_{P, n} &= 0 && \text{for } 0 < n < 1, \\ a_{P, n} &= |\kappa_P|^r + \mathcal{O}(|\kappa_P|^{r-1}) && \text{for } n = 1 + p^{-1}, \\ a_{P, n} &= 0 && \text{for } 1 + p^{-1} < n < 2, \\ a_{P, n} &= \mathcal{O}(E^l |\kappa_P|^{lr-1}) && \text{for } l \leq n < l + lp^{-1} \text{ with } l = 1, \dots, p-1, \\ a_{P, n} &= \mathcal{O}(E^l |\kappa_P|^{lr-1}) && \text{for } l \leq n < l + 1 \text{ with } l = p, p+1, \dots, \\ a_{P, n} &= \mathcal{O}(E^l |\kappa_P|^{lr}) && \text{for } l + lp^{-1} \leq n < l + 1 \text{ with } l = 2, \dots, p-1. \end{aligned}$$

If $\mathfrak{g}[p]$ is abelian, then:

$$\begin{aligned} a_{P, n} &= 1 && \text{for } n = 0, \\ a_{P, n} &= 0 && \text{for } 0 < n < 1, \\ a_{P, n} &= |\kappa_P|^r + \mathcal{O}(1) && \text{for } n = 1, \\ a_{P, n} &= 0 && \text{for } 1 < n < 2, \\ a_{P, n} &= \mathcal{O}(E^l |\kappa_P|^{lr}) && \text{for } l \leq n < l + 1 \text{ with } l = 2, 3, \dots \end{aligned}$$

We now fix a real $\varepsilon > 0$ satisfying $\varepsilon < \frac{1}{2}$ and, if $\mathfrak{g}[p]$ is non-abelian (in which case we have assumed $r \leq p-1$), also satisfying $\varepsilon < \frac{p-r}{p+1}$. We assume that $q \geq E^{1/\varepsilon}$, so that $E \leq |\kappa_P|^\varepsilon$ for all places P (note that any q works if $E = 1$).

If $\mathfrak{g}[p]$ is non-abelian: The estimates above show that we can take the following values of b_n, e_n, k_n in order to apply Lemma 5.4:

n	$0 < n < 1$ or $1 + p^{-1} < n < 2$	$1 + p^{-1}$	$l + lp^{-1} \leq n < l + 1$ with $2 \leq l < p$	$l \leq n < l + lp^{-1}$ with $1 \leq l < p$ or $l \leq n < l + 1$ with $l \geq p$
b_n	0	1	0	0
e_n	0	r	0	0
k_n	$-\infty$	$r - 1$	$l\varepsilon + lr$	$l\varepsilon + lr - 1$

In the notation of Lemma 5.4, we have $S = \{1 + p^{-1}\}$, $A = \frac{r+1}{1+p^{-1}}$, $B = 1$ and $M = 1 + p^{-1}$. We verify Inequality (5.4):

$$\begin{aligned} \frac{l\varepsilon + lr + 1}{l + lp^{-1}} &\leq \frac{\varepsilon + r + 1/2}{1 + p^{-1}} < \frac{r + 1}{1 + p^{-1}} = A && \text{for } 2 \leq l < p, \\ \frac{l\varepsilon + lr}{l} = \varepsilon + r &< \frac{p-r}{p+1} + r = \frac{r+1}{1+p^{-1}} = A && \text{for } l \geq 1. \end{aligned}$$

If $\mathfrak{g}[p]$ is abelian: The estimates above show that we can take the following values of b_n, e_n, k_n in order to apply Lemma 5.4:

n	$0 < n < 1$	1	$1 < n < 2$	$l \leq n < l + 1$ with $l \geq 2$
b_n	0	1	0	0
e_n	0	r	0	0
k_n	$-\infty$	0	$-\infty$	$l\varepsilon + lr$

We have $S = \{1\}$, $A = r + 1$, $B = 1$ and $M = 1$. We verify Inequality (5.4):

$$\frac{l\varepsilon + lr + 1}{l} \leq \varepsilon + r + \frac{1}{2} < r + 1 = A \quad \text{for } l \geq 2. \quad \square$$

6. GROUPS OF EXPONENT p : THE EXAMPLE OF HEISENBERG GROUPS

When \mathfrak{g} has exponent p , our main result (Theorem 1.2) only applies to finitely many p -groups due to the assumption that $|\mathfrak{g}| = |\mathfrak{g}[p]| \leq p^{p-1}$. To illustrate how this restriction may be overcome, we are going to deal with the infinite family of Heisenberg groups $H_k(\mathbb{F}_p)$ of exponent p .

The section is organized as follows: in Subsections 6.1 and 6.2, we review the simpler form taken by nilpotent Artin–Schreier theory and by the equations of Definition 3.9 when focusing on groups of exponent p . In particular, we show how a stronger version of Corollary 4.13 follows immediately from the equations (Proposition 6.2). We then introduce generalized Heisenberg groups in Subsection 6.3, we obtain estimates for the number of elements of $H_k(\mathbb{F}_q)$ which commute with (part of) their Frobenius orbit in Subsection 6.4, and we use these estimates to obtain asymptotics for the number of $H_k(\mathbb{F}_p)$ -extensions of function fields in Subsections 6.5 and 6.6, proving our main theorem (Theorem 6.7, which is Theorem 1.3).

6.1. Nilpotent Artin–Schreier theory in exponent p . When we focus only on p -groups G of exponent p and of nilpotency class at most 2 (corresponding to a Lie \mathbb{F}_p -algebra \mathfrak{g}), we do not need Witt vectors at all, as we shall briefly explain.

Let $\mathfrak{g} \neq 0$ be a finite Lie \mathbb{F}_p -algebra of nilpotency class ≤ 2 . For every field of characteristic p , we have a bijection $\text{orb}: H^1(\Gamma_F, G) \xrightarrow{\sim} \mathfrak{g} \otimes F//_F$, where $\mathfrak{g} \otimes F//_F$ is the set of orbits of $\mathfrak{g} \otimes_{\mathbb{F}_p} F$ under the $(\mathfrak{g} \otimes F, \circ)$ -action given by $g.m := \sigma(g) \circ m \circ (-g)$. In other words, compared to previous sections, we can ignore every coordinate of every Witt vector besides the first, and we do not need to take the perfect closure of the base field (see Lemma 2.9). All proofs are similar to the proofs of Section 2, but are largely simplified.

When we have fixed a finite field κ , we let \mathcal{D}^0 (resp. \mathcal{D}) be the κ -linear subspace of $\kappa((\pi))$ spanned by the elements π^{-a} for $a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}$ (resp. for $a \in \mathbb{N} \setminus p\mathbb{N}$). Then, the obvious analogues of all results from Section 3 hold, with simplified proofs. In particular, we have a bijection $\alpha^0: \mathfrak{g} \otimes F//_F \xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{D}^0//_\kappa$ and a $|\mathfrak{g} \otimes \kappa|$ -to-one surjection $\alpha: \mathfrak{g} \otimes F//_F \xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{D}//_\kappa$.

6.2. The distribution of last jumps in exponent p . Let as above \mathfrak{g} be a non-zero finite Lie \mathbb{F}_p -algebra of nilpotency class ≤ 2 and let $r := \dim_{\mathbb{F}_p} \mathfrak{g}$. By Theorem 3.20, the condition that an element $D \in \mathfrak{g} \otimes \mathcal{D}$ satisfies $\text{lastjump}(D) < v$ is equivalent to the equations given in Definition 3.9. Moreover, in this setting, these equations take a much simpler form:

Corollary 6.1. *Consider an element $D \in \mathfrak{g} \otimes \mathcal{D}$, written as $D = \sum_{b \in \mathbb{N} \setminus p\mathbb{N}} D_b \pi^{-b}$ with $D_b \in \mathfrak{g} \otimes \kappa$. Then, D satisfies $\text{lastjump}(D) < v$ if and only if the following equations hold for all $b \in \mathbb{N} \setminus p\mathbb{N}$:*

$$D_b = -(2b)^{-1} \sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}: \\ b = a_1 + a_2, \\ a_1 < v, a_2 < v}} a_1 [D_{a_1}, D_{a_2}] \quad \text{if } b \geq v \quad (6.1)$$

$$0 = \sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}: \\ bp^{\mu v(b)} = a_1 + a_2, \\ a_1 < v, a_2 < v}} a_1 [D_{a_1}, D_{a_2}] \quad \text{if } b < v \quad (6.2)$$

$$0 = \sum_{\substack{a_1, a_2 \in \mathbb{N} \setminus p\mathbb{N}: \\ b = a_1 p^i + a_2, \\ a_1 < v, a_2 < v}} a_1 [\sigma^i(D_{a_1}), D_{a_2}] \quad \text{for all } i > 0 \text{ such that } bp^{-i} \geq v. \quad (6.3)$$

This simpler form of the equations can be used to directly obtain a stronger form of Corollary 4.13/Proposition 4.15 when the exponent is p (with a much simpler proof):

Proposition 6.2. *For all $v > 0$, we have the upper bound*

$$\begin{aligned} |\{D \in \mathfrak{g} \otimes \mathcal{D} \mid \text{lastjump}(D) < v\}| &= \left| \left\{ (D_a)_{p|a < v} \in (\mathfrak{g} \otimes \kappa)^{[v] - \lceil \frac{v}{p} \rceil} \mid \text{Equations (6.2) and (6.3) hold} \right\} \right| \\ &\leq |\mathfrak{g} \otimes \kappa|^{[v] - \lceil \frac{v}{p} \rceil} = |\kappa|^{r([v] - \lceil \frac{v}{p} \rceil)}. \end{aligned}$$

Proof. The elements D_b for $b \geq v$ are uniquely expressed in terms of the elements D_b for $b < v$ using Equation (6.1), and they never appear in Equations (6.2) and (6.3). This implies the first equality. The following inequality is clear. \square

In order to apply Lemma 5.4 without assuming that r is small, we need to better understand the distribution of extensions with small last jump, and hence study the sets $A_m(\kappa)$ appearing in Remark 4.7 (as \mathfrak{g} has exponent p , we have $\mathfrak{g}[p] = \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbb{Z}_p} W(\kappa) = \mathfrak{g} \otimes_{\mathbb{F}_p} \kappa$). Since the sizes of these sets heavily depend on the Lie algebra (we are essentially counting elements commuting with their Frobenii), a unified statement for the asymptotics seems out of reach.³

6.3. Higher Heisenberg groups and their Lie algebras. Let $p \neq 2$ and $k \geq 1$. For us, the Heisenberg group $H_k(\mathbb{F}_p)$ is the group defined by the matrix representation⁴

$$H_k(\mathbb{F}_p) := \left\{ \begin{pmatrix} 1 & \vec{a}^t & c \\ 0 & I_k & \vec{b} \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_{k+2}(\mathbb{F}_p) \mid \vec{a}, \vec{b} \in \mathbb{F}_p^k, c \in \mathbb{F}_p \right\}. \quad (6.4)$$

Equivalently, it is the unique group (up to isomorphism) with exponent p , center $\mathbb{Z}/p\mathbb{Z}$, nilpotency class 2, and order p^{2k+1} . We denote the corresponding Lie \mathbb{F}_p -algebra by \mathfrak{h}_k , which as an \mathbb{F}_p -vector space can be decomposed as $\mathbb{F}_p \oplus (\mathbb{F}_p^k)^2$ (the \mathbb{F}_p -factor is the center, corresponding to the coefficient c , and the two \mathbb{F}_p^k -factors correspond respectively to \vec{a} and \vec{b}). The Lie bracket of \mathfrak{h}_k is determined by the induced map $(\mathfrak{h}_k/Z(\mathfrak{h}_k))^2 \rightarrow Z(\mathfrak{h}_k)$, which is a non-degenerate \mathbb{F}_p -bilinear alternating form $f_k: (\mathbb{F}_p^k)^2 \rightarrow \mathbb{F}_p$. Concretely, if (\vec{a}, \vec{b}) and (\vec{a}', \vec{b}') are two elements of $(\mathbb{F}_p^k)^2$, one can check that

$$f_k((\vec{a}, \vec{b}), (\vec{a}', \vec{b}')) = \vec{a} \cdot \vec{b}' - \vec{b} \cdot \vec{a}' = \sum_{i=1}^k (a_i b'_i - b_i a'_i).$$

(In fact, f_k is the only non-degenerate bilinear alternating form up to automorphisms of $(\mathbb{F}_p^k)^2$.)

³The main dependence in the Lie algebra \mathfrak{g} certainly comes from its ‘‘commuting variety’’ $C \subseteq (\mathbb{A}_{\mathbb{F}_p}^r)^2$, cut out by the equations corresponding to $[x, y] = 0$. For instance, for $A_1(\mathbb{F}_q)$, we are counting \mathbb{F}_q -points of the intersection of C with the graph of σ . Interpreting this as an intersection number, and using the Grothendieck–Lefschetz trace formula (following [SV22, HHYZ24]) might relate this number to the geometry of C .

⁴Be aware that this generalization of the Heisenberg group is distinct from the generalization considered in [Mü22] (where the rank of the center is not always 1).

6.4. Elements commuting with their Frobenii. For any finite field \mathbb{F}_q of characteristic p and any $m \geq 0$, extend the bilinear form f_k to \mathbb{F}_q^{2k} and define the set

$$A_{k,m}(\mathbb{F}_q) := \left\{ x \in \mathbb{F}_q^{2k} \mid f_k(\sigma^i(x), x) = 0 \text{ for } i \in \{1, \dots, m\} \right\}. \quad (6.5)$$

Clearly, $A_{k,0}(\mathbb{F}_q) \supseteq A_{k,1}(\mathbb{F}_q) \supseteq A_{k,2}(\mathbb{F}_q) \supseteq \dots$. The definition of $A_{k,m}(\mathbb{F}_q)$ is motivated by its appearance in Remark 4.7, which implies the following equality when the base field is $\mathbb{F}_q((\pi))$:

$$\begin{aligned} |\{D \in \mathfrak{h}_k \otimes \mathcal{D} \mid \text{lastjump}(D) < 1 + p^{-m}\}| &= |Z(\mathfrak{h}_k) \otimes \mathbb{F}_q| \cdot |A_{k,m}(\mathbb{F}_q)| \\ &= q \cdot |A_{k,m}(\mathbb{F}_q)|. \end{aligned} \quad (6.6)$$

We say that a subspace W of \mathbb{F}_q^{2k} is *isotropic* if $f_k(x, y) = 0$ for all $x, y \in W$. Note that all maximal isotropic subspaces of \mathbb{F}_q^{2k} are k -dimensional.

Lemma 6.3 (Large m). *We have*

$$A_{k,m}(\mathbb{F}_q) = \bigcup_{\substack{W \subseteq \mathbb{F}_p^{2k} \\ \text{isotropic}}} (W \otimes_{\mathbb{F}_p} \mathbb{F}_q) \quad \text{for } m \geq k.$$

Proof.

- (\supseteq) Let $W \subseteq \mathbb{F}_p^{2k}$ be isotropic and let $x \in W \otimes \mathbb{F}_q$. Then, $\sigma^i(x) \in W \otimes \mathbb{F}_q$ for all $i \in \mathbb{Z}$. Hence, $f_k(\sigma^i(x), x) = 0$ for all $i \in \mathbb{Z}$.
- (\subseteq) Let $x \in A_{k,m}(\mathbb{F}_q)$. We consider the vector space W' over \mathbb{F}_q generated by $x, \sigma(x), \dots, \sigma^k(x)$. By assumption, $f_k(\sigma^j(x), \sigma^i(x)) = \sigma^i(f_k(\sigma^{j-i}(x), x)) = 0$ for all $0 \leq i < j \leq k$, so W' is isotropic. The maximal isotropic subspaces of \mathbb{F}_q^{2k} are k -dimensional, so $\dim_{\mathbb{F}_q}(W') \leq k$. Hence, the $k+1$ vectors $x, \sigma(x), \dots, \sigma^k(x)$ are linearly dependent over \mathbb{F}_q , so $\sigma(W') = W'$. By Galois descent for vector spaces, $W' = W \otimes_{\mathbb{F}_q} \mathbb{F}_p$ for some isotropic \mathbb{F}_p -subspace $W = (W')^\sigma$ of \mathbb{F}_p^{2k} . The claim follows as x lies in $W' = W \otimes \mathbb{F}_q$. \square

Lemma 6.4 (Counting maximal isotropic subspaces). *There are exactly $\prod_{i=1}^k (p^i + 1)$ maximal (i.e., k -dimensional) isotropic subspaces of \mathbb{F}_p^{2k} .*

Proof. We count tuples (b_1, \dots, b_k) of linearly independent vectors such that $f_k(b_i, b_j) = 0$ for all $1 \leq i, j \leq k$. After choosing linearly independent vectors b_1, \dots, b_i , there are exactly p^{2k-i} vectors b_{i+1} such that $f_k(b_{i+1}, b_j) = 0$ for $j = 1, \dots, i$, as the orthogonal complement of the i -dimensional space $\langle b_1, \dots, b_i \rangle$ has dimension $2k - i$. Exactly p^i of these vectors b_{i+1} lie in $\langle b_1, \dots, b_i \rangle$, so that there are $p^{2k-i} - p^i$ choices for b_{i+1} . Hence, the number of tuples (b_1, \dots, b_k) as above is

$$\prod_{i=0}^{k-1} (p^{2k-i} - p^i) = \prod_{i=0}^{k-1} p^i (p^{2(k-i)} - 1).$$

Each maximal isotropic subspace W , being k -dimensional, has exactly

$$|\text{GL}_k(\mathbb{F}_p)| = \prod_{i=0}^{k-1} (p^k - p^i) = \prod_{i=0}^{k-1} p^i (p^{k-i} - 1)$$

bases (b_1, \dots, b_k) . Hence, the number of maximal isotropic subspaces is

$$\prod_{i=0}^{k-1} \frac{p^i (p^{2(k-i)} - 1)}{p^i (p^{k-i} - 1)} = \prod_{i=0}^{k-1} (p^{k-i} + 1) = \prod_{i=1}^k (p^i + 1). \quad \square$$

Lemma 6.5 (Small m). *We have $|A_{k,m}(\mathbb{F}_q)| = q^{2k-m} + \mathcal{O}_k(q^k)$ for $0 \leq m < k$.*

Proof. We handle the conditions $f_k(\sigma^i(x), x) = 0$ for $i = 1, \dots, m$ using the following sum over the q^m characters χ of the (additive) group \mathbb{F}_q^m :

$$|A_{k,m}(\mathbb{F}_q)| = \frac{1}{q^m} \sum_{\chi} \sum_{x \in (\mathbb{F}_q^k)^2} \chi(f_k(\sigma(x), x), \dots, f_k(\sigma^m(x), x)).$$

Writing $x = (\vec{a}, \vec{b})$ with $\vec{a}, \vec{b} \in \mathbb{F}_q^k$, we have by definition $f_k(\sigma^i(x), x) = \sigma^i(\vec{a}) \cdot \vec{b} - \sigma^i(\vec{b}) \cdot \vec{a} = \sum_{j=1}^k (\sigma^i(a_j)b_j - \sigma^i(b_j)a_j)$. We can then factor the inner sum to obtain

$$|A_{k,m}(\mathbb{F}_q)| = \frac{1}{q^m} \sum_{\chi} \left(\sum_{a,b \in \mathbb{F}_q} \chi(\sigma(a)b - \sigma(b)a, \dots, \sigma^m(a)b - \sigma^m(b)a) \right)^k.$$

Using the non-degenerate trace form $(x, y) \mapsto \text{Tr}_{\mathbb{F}_q|\mathbb{F}_p}(xy)$, we can identify the characters χ of \mathbb{F}_q^m with vectors $t \in \mathbb{F}_q^m$, so that

$$|A_{k,m}(\mathbb{F}_q)| = \frac{1}{q^m} \sum_{t \in \mathbb{F}_q^m} F(t)^k,$$

where

$$F(t) := \sum_{a,b \in \mathbb{F}_q} e_p \left(\sum_{i=1}^m \text{Tr} \left(t_i (\sigma^i(a)b - \sigma^i(b)a) \right) \right)$$

with $e_p(x \bmod p) := e^{2\pi i x/p}$. For $t = 0$ (corresponding to the trivial character), we clearly have $F(0) = q^2$. For $t \neq 0$, we split up the sum $F(t)$ into sub-sums $F(t, b)$ according to the choice of $b \in \mathbb{F}_q$. Clearly, $F(t, 0) = q$. For $b \neq 0$, the substitution $r := a/b$ shows

$$\begin{aligned} F(t, b) &= \sum_{r \in \mathbb{F}_q} e_p \left(\sum_{i=1}^m \text{Tr} \left(t_i \sigma^i(b)b \cdot (\sigma^i(r) - r) \right) \right) \\ &= \sum_{r \in \mathbb{F}_q} e_p \left(\sum_{i=1}^m \left(\text{Tr}(t_i \sigma^i(b)b \sigma^i(r)) - \text{Tr}(t_i \sigma^i(b)br) \right) \right) \\ &= \sum_{r \in \mathbb{F}_q} e_p \left(\sum_{i=1}^m \left(\text{Tr}(\sigma^{-i}[t_i \sigma^i(b)b]r) - \text{Tr}(t_i \sigma^i(b)br) \right) \right) \quad \text{as the trace is } \sigma\text{-invariant} \\ &= \sum_{r \in \mathbb{F}_q} e_p \left(\text{Tr} \left(\sum_{i=1}^m (\sigma^{-i}[t_i \sigma^i(b)b] - t_i \sigma^i(b)b) \cdot r \right) \right). \end{aligned}$$

By orthogonality of characters of \mathbb{F}_q and non-degeneracy of the trace form, we obtain

$$F(t, b) = \begin{cases} q & \text{if } \sum_{i=1}^m (\sigma^{-i}[t_i \sigma^i(b)b] - t_i \sigma^i(b)b) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying the bijection σ^m to the condition $\sum_{i=1}^m (\sigma^{-i}[t_i \sigma^i(b)b] - t_i \sigma^i(b)b) = 0$ turns it into the following polynomial equation in the variable b :

$$\sum_{i=1}^m \left(\sigma^{m-i}(t_i)b^{p^m+p^{m-i}} - \sigma^m(t_i)b^{p^{m+i}+p^m} \right) = 0.$$

For any fixed $t \neq 0$, the left-hand side is a non-zero polynomial in b of degree at most $p^{2m} + p^m$. Hence, for any $t \neq 0$, there are at most $p^{2m} + p^m = \mathcal{O}(1)$ values $b \in \mathbb{F}_q^\times$ with $F(t, b) \neq 0$; as seen above, we have $F(t, b) = q$ in this case. We conclude that $F(t) = \sum_{b \in \mathbb{F}_q} F(t, b) = q + \sum_{b \in \mathbb{F}_q^\times} F(t, b) = \mathcal{O}(q)$ for all $t \neq 0$. Combining this with the fact that $F(0) = q^2$, we obtain

$$|A_{k,m}(\mathbb{F}_q)| = \frac{1}{q^m} \left(q^{2k} + \mathcal{O}(q^{m+k}) \right) = q^{2k-m} + \mathcal{O}(q^k). \quad \square$$

6.5. Local counting. In this subsection, we fix a finite field κ , and we use the estimates of the sizes of the sets $A_{k,m}(\kappa)$ obtained in the previous subsection in order to estimate the distribution of $H_k(\mathbb{F}_p)$ -extensions of the local function field $\kappa((\pi))$ (cf. Remark 4.7 and Lemma 4.1). More precisely, we prove the following lemma, in the spirit of Theorem 4.2:

Lemma 6.6. *Consider the local field $\mathfrak{F} = \kappa((\pi))$. For any $v \geq 0$, let $N(= v)$, $N(< v)$, $N(\leq v)$ be the number of $D \in \mathfrak{g} \otimes \mathcal{D}$ such that $\text{lastjump}(D) = v$, $\text{lastjump}(D) < v$, or $\text{lastjump}(D) \leq v$, respectively. We have:*

- (a) $N(= 0) = N(< 1) = 1$.
- (b) $N(\leq 1) = N(< 1 + p^{-k}) = \prod_{i=1}^k (p^i + 1) \cdot |\kappa|^{k+1} \cdot (1 + \mathcal{O}_k(|\kappa|^{-1}))$.
- (c) $N(\leq 1 + p^{-m-1}) = N(< 1 + p^{-m}) = |\kappa|^{2k+1-m} \cdot (1 + \mathcal{O}_k(|\kappa|^{-1}))$ for $0 \leq m < k$.
- (d) $N(< l + 1) = \mathcal{O}_k(|\kappa|^{(l-\lfloor l/p \rfloor)(2k+1)})$ for $l \geq 0$.
- (e) $N(< l + p^{-m}) = \mathcal{O}_k(|\kappa|^{l(2k+1-m)} + |\kappa|^{(l-1)(2k+1)+1})$ for $2 \leq l < p$ and $1 \leq m \leq k$.
- (f) $N(< l + lp^{-m}) = \mathcal{O}_k(|\kappa|^{l(2k+2-m)-1} + |\kappa|^{(l-1)(2k+1)+1})$ for $2 \leq l < p$ and $1 \leq m \leq k$.

Proof. Let π be the projection $\mathfrak{g} = \mathbb{F}_p^{2k} \oplus \mathbb{F}_p \rightarrow \mathbb{F}_p^{2k}$.

- (a) See Theorem 4.2 (a).
- (b) Equation (6.6) implies that $N(< 1 + p^{-m}) = |\kappa| \cdot |A_{k,m}(\kappa)|$, which by Lemma 6.3 does not depend on m as soon as $m \geq k$. Therefore, $N(\leq 1) = N(< 1 + p^{-k}) = |\kappa| \cdot |A_{k,k}(\kappa)|$. The claim then follows using Lemmas 6.3 and 6.4.
- (c) By Proposition 4.6 and Equation (6.6), we have $N(\leq 1 + p^{-m-1}) = N(< 1 + p^{-m}) = |\kappa| \cdot |A_{k,m}(\kappa)|$. The claim then follows using Lemma 6.5.
- (d) See Proposition 6.2 (we have $l + 1 - \lceil \frac{l+1}{p} \rceil = l - \lfloor \frac{l}{p} \rfloor$, and $r = 2k + 1$ in this case).
- (e) By Corollary 4.4 and Proposition 4.9 (i), $N(< l + p^{-m})$ is at most the number of tuples (D_1, \dots, D_l) of elements of $\mathfrak{g} \otimes \kappa$ such that $[\sigma^i(D_l), D_a] = 0$ for $i = 1, \dots, m$ and $a = 1, \dots, l$. We will first pick D_l and then D_1, \dots, D_{l-1} .

Let $V_i(D_l)$ be the κ -span of $\sigma(\pi(D_l)), \dots, \sigma^i(\pi(D_l))$. If $V_i(D_l) = V_{i+1}(D_l)$ for some i , then $V_i(D_l)$ is stable under σ and hence defined over \mathbb{F}_p by Galois descent for vector spaces; this implies that $V_j(D_l) = V_i(D_l)$ for all $j \geq i$. Let $V(D_l) := \bigcup_{i \geq 1} V_i(D_l)$, which is defined over \mathbb{F}_p for the same reason. Let $d(D_l) := \dim(V(D_l))$. We then have $\dim(V_i(D_l)) = \min(i, d(D_l))$.

The conditions $[\sigma^i(D_l), D_a] = 0$ for $i = 1, \dots, m$ mean that $\pi(D_a)$ has to lie in the orthogonal complement of $V_m(D_l)$ with respect to the alternating bilinear form f_k . This orthogonal complement has dimension $2k - \dim(V_m(D_l)) = 2k - \min(m, d(D_l))$, so the number of valid D_a is $|\kappa|^{2k+1-\min(m, d(D_l))}$ for $a = 1, \dots, l - 1$.

Focusing first on tuples (D_1, \dots, D_l) with $d(D_l) \geq m$, we bound the number of D_l such that $[\sigma^i(D_l), D_l] = 0$ for $i = 1, \dots, m$ using Lemmas 6.3, 6.4 and 6.5: the number of such D_l is $\mathcal{O}_k(|\kappa|^{2k+1-m})$, so the total number of valid tuples (D_1, \dots, D_l) with $d(D_l) \geq m$ is $\mathcal{O}_k(|\kappa|^{(l-1)(2k+1-m)+(2k+1-m)}) = \mathcal{O}_k(|\kappa|^{l(2k+1-m)})$.

Now, we fix some $d < m$ and focus on the case $d(D_l) = d$. In this case, we will have more choices for D_1, \dots, D_{l-1} . However, we will have fewer choices for D_l , as $\pi(D_l)$ must lie in the d -dimensional subspace $V(D_l)$, which is defined over \mathbb{F}_p . For any of the $\mathcal{O}_k(1)$ d -dimensional subspaces V of κ^{2k} defined over \mathbb{F}_p , there are $|\kappa|^{d+1}$ choices of D_l such that $\pi(D_l) \in V$, so there are at most $|\kappa|^{d+1}$ choices of D_l such that $V(D_l) = V$. Hence, for any given dimension d , we have $\mathcal{O}_k(|\kappa|^{(l-1)(2k+1-d)+(d+1)})$ valid tuples (D_1, \dots, D_l) with $d(D_l) = d$. The exponent $(l-1)(2k+1-d) + (d+1)$ is maximal for $d = 0$. Thus, the number of valid tuples (D_1, \dots, D_l) with $d(D_l) < m$ is $\mathcal{O}_k(|\kappa|^{(l-1)(2k+1)+1})$.

- (f) By Corollary 4.4 and Proposition 4.9 (ii), $N(< l + lp^{-m})$ is at most the number of tuples (D_1, \dots, D_l) of elements of $\mathfrak{g} \otimes \kappa$ such that $[\sigma^i(D_l), D_a] = 0$ for $i = 1, \dots, m - 1$ and $a = 1, \dots, l - 1$ and such that $[\sigma^i(D_l), D_l] = 0$ for $i = 1, \dots, m$. We distinguish the same two types of tuples as before:

If $d(D_l) \geq m$, then $V_{m-1}(D_l)$ has dimension $m - 1$, and thus the number of valid tuples with $d(D_l) \geq m$ is $\mathcal{O}_k(|\kappa|^{(l-1)(2k+1-(m-1))+(2k+1-m)}) = \mathcal{O}_k(|\kappa|^{l(2k+2-m)-1})$.

If $d(D_l) < m$, then $V_{m-1}(D_l) = V_m(D_l)$, so the number of valid tuples with $d(D_l) < m$ is $\mathcal{O}_k(|\kappa|^{(l-1)(2k+1)+1})$ as before. \square

6.6. Global asymptotics. Define the following numbers:

$$\begin{aligned} n(m) &:= 1 + p^{-m-1}, & e'_m &:= 2k + 1 - m, & b'_m &:= 1 & \text{for } 0 \leq m \leq k-1, \\ n(k) &:= 1, & e'_k &:= k + 1, & b'_k &:= \prod_{i=1}^k (p^i + 1). \end{aligned}$$

Define $A := \max\{\frac{e'_m+1}{n(m)} \mid 0 \leq m \leq k\}$ and $S' = \{0 \leq m \leq k \mid \frac{e'_m+1}{n(m)} = A\}$. Let $B := \sum_{m \in S'} b'_m$, and let M be the least common integer multiple of the rational numbers b'_m for $m \in S'$.

Theorem 6.7 (cf. Theorem 1.3). *There is a function $C : \mathbb{Q}/M\mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ with $C(0) \neq 0$, such that for rational $N \rightarrow \infty$, we have*

$$\sum_{\substack{K \in \text{ÉtExt}(H_k(\mathbb{F}_p), F): \\ \text{lastjump}(K)=N}} \frac{1}{|\text{Aut}(K)|} = C(N \bmod M) \cdot q^{AN} N^{B-1} + o(q^{AN} N^{B-1}).$$

Proof. We apply Lemma 5.4 just as in Subsection 5.3 (in the proof of Theorem 1.2), and with the same notation, but using the following estimates arising from Lemma 6.6:

$$\begin{aligned} a_{P,n} &= 1 && \text{for } n = 0, \\ a_{P,n} &= 0 && \text{for } 0 < n < 1, \\ a_{P,n} &= \prod_{i=1}^k (p^i + 1) \cdot |\kappa_P|^{k+1} + \mathcal{O}(|\kappa_P|^k) && \text{for } n = 1, \\ a_{P,n} &= 0 && \text{for } 1 < n < 1 + p^{-k}, \\ a_{P,n} &= |\kappa_P|^{2k+1-m} + \mathcal{O}(|\kappa_P|^{2k-m}) && \text{for } n = 1 + p^{-m-1} \text{ with } 0 \leq m \leq k-1, \\ a_{P,n} &= 0 && \text{for } 1 + p^{-m-1} < n < 1 + p^{-m} \\ &&& \text{with } 0 \leq m \leq k-1, \\ a_{P,n} &= \mathcal{O}\left(|\kappa_P|^{\max(l(k+1), (l-1)(2k+1)+1)}\right) && \text{for } l \leq n < l + p^{-k} \text{ with } 2 \leq l \leq p-1, \\ &&& \text{[Case I]} \\ a_{P,n} &= \mathcal{O}\left(|\kappa_P|^{\max(l(2k+2-m)-1, (l-1)(2k+1)+1)}\right) && \text{for } l + p^{-m} \leq n < l + lp^{-m} \\ &&& \text{with } 2 \leq l \leq p-1 \text{ and } 1 \leq m \leq k, \text{ [Case II]} \\ a_{P,n} &= \mathcal{O}\left(|\kappa_P|^{\max(l(2k+1-m), (l-1)(2k+1)+1)}\right) && \text{for } l + lp^{-m-1} \leq n < l + p^{-m} \\ &&& \text{with } 2 \leq l \leq p-1 \text{ and } 0 \leq m \leq k-1, \\ &&& \text{[Case III]} \\ a_{P,n} &= \mathcal{O}\left(|\kappa_P|^{(l-\lfloor l/p \rfloor)(2k+1)}\right) && \text{for } l \leq n < l+1 \text{ with } p \leq l. \text{ [Case IV]} \end{aligned}$$

The numbers A, B, M defined above are the same as the numbers A, B, M defined in Lemma 5.4. Inequality (5.4) is verified as follows:

For [Case I] (first argument of max):

$$\frac{l(k+1)+1}{l} < k+2 \leq A \quad \text{for } 2 \leq l \leq p-1,$$

For [Case II] (first argument of max):

$$\frac{l(2k+2-m)}{l+p^{-m}} < \left\{ \begin{array}{ll} \frac{2k+2-m}{1+p^{-m-1}} & \text{if } m \leq k-1, \\ k+2 & \text{if } m = k \end{array} \right\} \leq A \quad \text{for } 2 \leq l \leq p-1 \text{ and } 1 \leq m \leq k,$$

For [Case III] (first argument of max):

$$\frac{l(2k+1-m)+1}{l+lp^{-m-1}} < \frac{2k+2-m}{1+p^{-m-1}} \leq A \quad \text{for } 2 \leq l \leq p-1 \text{ and } 0 \leq m \leq k-1,$$

For [Cases I–III] (second argument of max):

$$\frac{(l-1)(2k+1)+2}{l} < k+2 \leq A \quad \text{for } l=2,$$

$$\begin{aligned} \frac{(l-1)(2k+1)+2}{l} &\leq \left(1 - \frac{1}{l}\right)(2k+2) \\ &< \left(1 - \frac{1}{p}\right)(2k+2) < \frac{2k+2}{1+\frac{1}{p}} \leq A \end{aligned} \quad \text{for } 3 \leq l \leq p-1,$$

For [Case IV]:

$$\begin{aligned} \frac{(l - \lfloor l/p \rfloor)(2k+1)+1}{l} &\leq \frac{l - \lfloor l/p \rfloor + 1/3}{l}(2k+1) \\ &\leq \frac{2p-2+1/3}{2p-1}(2k+1) < \left(1 - \frac{1}{p^2}\right)(2k+1) \\ &< \left\{ \begin{array}{ll} \frac{2k+1}{1+p^{-2}} & \text{if } k \geq 2 \\ k+2 & \text{if } k = 1 \end{array} \right\} \leq A \end{aligned} \quad \text{for } p \leq l. \quad \square$$

Finally, we describe the integer B more concretely:

Proposition 6.8. *If $(p, k) = (3, 1)$, then $A = 3$ and $B = 5$. If $(p, k) = (3, \frac{3^{m+2}+2m-1}{4})$ for some $m \geq 0$, then $A = \frac{e'_m+1}{n(m)} = \frac{e'_{m+1}+1}{n(m+1)}$ and $B = 2$. In all other cases, $B = 1$.*

Proof. The case $(p, k) = (3, 1)$ is clear. Otherwise, at least one of the inequalities in

$$\frac{e'_0+1}{n(0)} = \frac{2k+2}{1+p^{-1}} \geq \frac{3}{4}(2k+2) = \frac{3}{2}(k+1) \geq k+2 = \frac{e'_k+1}{n(k)},$$

is strict, so that $k \notin S'$, and then

$$A = \max \left\{ \frac{e'_m+1}{n(m)} \mid 0 \leq m \leq k-1 \right\} = \max \left\{ \frac{2k+2-m}{1+p^{-m-1}} \mid 0 \leq m \leq k-1 \right\}.$$

Consider the derivative

$$\frac{d}{dm} \frac{2k+2-m}{1+p^{-m-1}} = \frac{-p^{m+1}-1+(2k+2-m)\log p}{p^{m+1}(1+p^{-m-1})^2}.$$

The denominator is positive, and the numerator is a strictly decreasing function of m . Hence, there is a single threshold $0 \leq r \leq k-1$ such that $\frac{2k+2-m}{1+p^{-m-1}}$ is strictly increasing for real $0 \leq m \leq r$ and strictly decreasing for real $r \leq m \leq k-1$.

This implies that $\frac{e'_m+1}{n(m)}$ can be maximal for at most two integers $0 \leq m \leq k-1$, and these integers would need to be consecutive. Conversely, if two consecutive integers produce the same value $\frac{e'_m+1}{n(m)}$, then this value must be A .

If $\frac{e'_m+1}{n(m)}$ is maximal for only one integer $0 \leq m \leq k-1$, then $B = 1$ as claimed, so assume $\frac{e'_m+1}{n(m)} = \frac{e'_{m+1}+1}{n(m+1)}$ with $0 \leq m \leq k-2$. This is equivalent to

$$\frac{2k+2-m}{1+p^{-m-1}} = \frac{2k+1-m}{1+p^{-m-2}},$$

so $1+p^{-m-2} = (2k+1-m)(p^{-m-1}-p^{-m-2})$, and hence $p^{m+2}+1 = (2k+1-m)(p-1)$. In particular, $p-1$ has to divide $p^{m+2}+1$, but $p^{m+2}+1 \equiv 2 \pmod{p-1}$, so $p = 3$. Plugging $p = 3$ back in and solving for k , we obtain $k = \frac{3^{m+2}+2m-1}{4}$. \square

CHART OF NOTATIONS

	Notation	Reference	Short description
Everywhere	p, σ, \wp		An odd prime, the absolute Frobenius, the map $\sigma - \text{id}$
	$\text{ÉtExt}(G, F)$	Subsection 2.1	The set of isomorphism classes of (étale) G -extensions of F
	$\text{lastjump}(K)$	Subsection 1.2	Last jump of an extension $K F$
	R^{perf}	Subsection 2.3	Perfect closure of a ring R
	$W(R), \text{Ver}$	Subsection 2.4	Witt vectors over R , the Verschiebung map
	$W_n(R)$	Subsection 2.4	Witt vectors of length n over R
Sections 3–6	G		A finite p -group of nilpotency class ≤ 2
	\mathfrak{g}		The finite Lie \mathbb{Z}_p -algebra associated to G
	\circ		The group law on \mathfrak{g} such that $G = (\mathfrak{g}, \circ)$
	\wp		The multiplicative Artin–Schreier map $x \mapsto \sigma(x) \circ (-x)$
	$g.m$		$\sigma(g) \circ m \circ (-g)$, for $g, m \in \mathfrak{g} \otimes W(F^{\text{perf}})$
	$G[p], \mathfrak{g}[p]$	Lemma 2.14	The p -torsion subgroup of G (resp. ideal of \mathfrak{g})
	$\mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$		The set of $(\mathfrak{g} \otimes W(F^{\text{perf}}), \circ)$ -orbits of $\mathfrak{g} \otimes W(F^{\text{perf}})$
	orb		The bijection $H^1(\Gamma_F, G) \xrightarrow{\sim} \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})}$
	$\text{lastjump}(D)$		Last jump of the G -extension of F associated to $D \in \mathfrak{g} \otimes W(F^{\text{perf}})$
Section 3 and Section 4	$\mathfrak{F}, \kappa, \pi$		A local function field of char. p , its residue field, a uniformizer
	$\tilde{\pi}$		The Teichmüller representative of π in $W(\mathfrak{F})$
	\mathcal{D}^0	Definition 3.2	$W(\kappa)$ -module spanned by $\tilde{\pi}^{-a}$ for $a \in \{0\} \cup \mathbb{N} \setminus p\mathbb{N}$
	$\mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)}$	Remark 3.5	Set of $(\mathfrak{g} \otimes W(\kappa), \circ)$ -orbits of $\mathfrak{g} \otimes \mathcal{D}^0$
	α^0	Theorem 3.6	The bijection $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})} \xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)}$
	$\mu_v(b)$	Equation (3.1)	Smallest integer $k \geq 0$ such that $b p^k \geq v$
	\mathcal{D}	Definition 3.14	$W(\kappa)$ -module spanned by $\tilde{\pi}^{-a}$ for $a \in \mathbb{N} \setminus p\mathbb{N}$
	pr	Definition 3.14	The projection $\mathcal{D}^0 \rightarrow \mathcal{D}$
	$g.D$	Proposition 3.16	$D - \frac{1}{2}[D, \sigma(g) + g]$, for $g \in \mathfrak{g} \otimes W(\kappa)$ and $D \in \mathfrak{g} \otimes \mathcal{D}$
	$\mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}$	Definition 3.17	Set of $(\mathfrak{g} \otimes W(\kappa), \circ)$ -orbits of $\mathfrak{g} \otimes \mathcal{D}$
	α	Definition 3.18	The surjection $\mathfrak{g} \otimes W(\mathfrak{F}^{\text{perf}}) //_{W(\mathfrak{F}^{\text{perf}})} \rightarrow \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}$
	$\text{lastjump}(D)$	Definition 3.24	Common last jump of all G -extensions associated to $D \in \mathfrak{g} \otimes \mathcal{D}$
Section 5	q, F		A power of p , the global function field $\mathbb{F}_q(T)$
	P, F_P		A place of F , the completion of F at P
	$\pi_P, \mathcal{D}_P, \alpha_P, \dots$		The associated local objects at P (cf. Section 3)
	α	Equation (5.1)	The map $\mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \rightarrow \prod'_P \mathfrak{g} \otimes \mathcal{D}_P //_{W(\kappa_P)}$
Section 6	$H_k(\mathbb{F}_p), \mathfrak{h}_k$	Subsection 6.3	Higher Heisenberg group, the corresponding Lie \mathbb{F}_p -algebra
	f_k	Subsection 6.3	The bilinear form $(\mathbb{F}_p^k)^2 \rightarrow \mathbb{F}_p$ inducing the Lie bracket of \mathfrak{h}_k
	$A_{k,m}(\mathbb{F}_q)$	Equation (6.5)	Set of $x \in (\mathbb{F}_q^k)^2$ such that $f_k(\sigma^i(x), x) = 0$ for $i = 1, \dots, m$

We also sum up in a diagram the three main bijections which we construct in this paper:

$$\text{ÉtExt}(G, F) \xleftarrow[\text{Lemma 2.1}]{\sim} H^1(\Gamma_F, G) \xrightarrow[\text{Theorem 2.19}]{\text{orb } \sim} \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \xrightarrow[\underbrace{\text{Theorem 3.6}}_{\text{case } F = \kappa(\pi) \text{ local}}]{\alpha^0 \sim} \mathfrak{g} \otimes \mathcal{D}^0 //_{W(\kappa)}.$$

When $F = \kappa(\pi)$ is a local field, there is also the surjection $\alpha: \mathfrak{g} \otimes W(F^{\text{perf}}) //_{W(F^{\text{perf}})} \rightarrow \mathfrak{g} \otimes \mathcal{D} //_{W(\kappa)}$ from Definition 3.18, which is finite-to-one (fibers have size $|\mathfrak{g} \otimes W(\kappa)|$), and all elements in a given fiber correspond to extensions with the same last jump (cf. Corollary 3.23).

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