A categorical perspective on splitting phenomena

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1 Splitting phenomena, categorified

Splitting phenomena. Let G be a group, and H a subgroup of G. It is a well-known fact that elements of H that are conjugate in G can fail to be conjugate in H, i.e. if c is a conjugacy class of G, the conjugation-invariant subset $c \cap H$ of H is not necessarily a conjugacy class.

In their work concerning the Cohen-Lenstra conjecture, Ellenberg, Venkatesh and Westerland noticed the usefulness of the *non-splitting property* (which I am about to give a slightly more general version of).

Definition 1. *Let G be a group and c a conjugation-invariant subset of G. We denote by D the set of conjugacy classes of G contained in c. We say that (G,c) is* non-splitting *if for every subgroup H of G that intersects every conjugacy class in D, and for every $\gamma \in D$, the subset $\gamma \cap H$ is a conjugacy class of H.*

The category ConjInv. Now let me introduce the category **ConjInv**. Its objects are couples (G,c) where G is a group and c is a conjugation-invariant set of G, and the morphisms between (H,c_H) and (G,c_G) are the morphisms $f : H \to G$ such that $f(c_H) \subseteq c_G$. The forgetful functor $P : \mathbf{ConjInv} \to \mathbf{Gp}$ given by $(G,c) \mapsto G$ is faithful.

If G is a group, we denote by $\mathbf{ConjInv}_G$ the category $P^{-1}(\{G\})$ of **ConjInv** with morphisms $(G,c) \to (G,c')$ each time $c \subseteq c'$ (this is a comma category).

Moreover if $f : H \to G$ is a morphism and $(H,c_H) \in \mathbf{ConjInv}_H$, there is an initial element among morphisms φ in **ConjInv** such that $P(\varphi) = f$ whose source is (H,c_H), given by the closure under conjugation of $f(c_H)$ in G. We denote by f_* the morphism so obtained.

Dually, if $f : H \to G$ is a morphism and $(G,c_G) \in \mathbf{ConjInv}_H$, there is an final element among morphisms φ in **ConjInv** such that $P(\varphi) = f$ whose target is (G,c_G), given by the intersection $f^{-1}(c_G) \cap H$. We denote by f^* the morphism so obtained.

Categorical view on splitting phenomena. Now assume $f : H \to G$ is an inclusion of groups.

What is $f_* f^*(G,c)$? It is the closure under conjugation of $c \cap H$. It is always contained in (G,c), and equal to (G,c) exactly when H intersects every conjugacy class of G contained in c. This is a reformulation of the hypothesis in Definition 1:

Definition 2. *We say that a morphism $f : H \to G$ is fully-intersecting at $(G,c) \in \mathbf{ConjInv}_G$ if $f_* f^*(G,c) = (G,c)$.*

Now, what does it mean for an element (H,c_H) to satisfy $f_*(H,c_H) = (G,c)$? It exactly means that c_H contains at least one element of each conjugacy class in c. If some class in c splits into many conjugacy classes in H, we can have an element of $\mathbf{ConjInv}_H$ below $(H, c \cap H)$

whose image by f_* is (G, c), by choosing either of the conjugacy classes of H that the class has split into. The non-splitting property for H is asking for this situation to be impossible, i.e. $(H, c \cap H)$ must be minimal among elements of $\mathbf{ConjInv}_H$ whose image by f_* is (G, c).

Definition 3. *A morphism $f : H \to G$ fully-intersecting at (G, c) is non-splitting at (G, c) if $f^*(G, c)$ is initial among elements $(H, c_H) \in \mathbf{ConjInv}_H$ such that $f_*(H, c_H) = (G, c)$.*

Definition 4. *An element $(G, c) \in \mathbf{ConjInv}$ is* non-splitting *if every monomorphism $H \to G$ which is fully-intersecting at (G, c) is non-splitting at (G, c).*

2 A general definition

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a category equipped with a faithful functor $P : \mathcal{D} \to \mathcal{C}$. If $x \in \mathcal{C}$, we denote by $\mathcal{D}_x$ the category of elements of $\mathcal{D}$ whose image by P is x, keeping only morphisms whose image by P are id_x.

We assume that for every $x' \in \mathcal{D}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(P(x), y)$ there is a morphism f_*, initial among such morphisms, whose source is x' and whose image by P is f.

Dually, we assume that for every $y' \in \mathcal{D}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(x, P(y))$ there is a morphism f^*, final among such morphisms, whose target is y' and whose image by P is f.

Definition 5.

- *A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ is fully-intersecting at $y' \in \mathcal{D}$ if $f_* f^* y' = y'$.*
- *A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ fully-intersecting at $y' \in \mathcal{D}$ is non-splitting at y' if $f^* y'$ is initial among elements z such that $f_* z = y'$.*
- *An element $y \in \mathcal{D}$ is non-splitting if every monomorphism into $P(y)$ which is fully-intersecting at y is non-splitting at y.*

Is this notion related in any way to a known categorical concept? What about the dual notions of co-fully-intersecting morphisms ($f^* f_* x' = x'$) and co-non-splitting morphisms ($f_* x'$ is final among elements z such that $f^* z = x'$)?

2.1 The case of topological spaces

Proposition 1. *If $\mathcal{C} = \mathbf{Set}$, $\mathcal{D} = \mathbf{Top}$ and P is the forgetful functor $\mathbf{Top} \to \mathbf{Set}$, then every element of $\mathbf{Top}$ is non-splitting.*

This illustrates a way in which the category of topological spaces is well-behaved. Was this fact observed?

Proof. Let $(Y, \mathcal{T})$ be a topological space and X be a subset of Y, with the inclusion $f : X \hookrightarrow Y$. Then $f^*(Y, \mathcal{T})$ is $(X, f^{-1}(T))$ and $f_*(X, \mathcal{T}')$ is Y equipped with the topology of sets U such that $f^{-1}(U) \in \mathcal{T}'$. Assume f to fully-intersecting at $(Y, \mathcal{T})$, i.e.:

$$f^{-1}(U) \in f^{-1}(\mathcal{T}) \Leftrightarrow U \in \mathcal{T}.$$

If $f_*(X, \mathcal{T}') = (Y, \mathcal{T})$, then:

$$f^{-1}(U) \in \mathcal{T}' \Leftrightarrow U \in \mathcal{T}.$$

Consider $V \in f^{-1}(\mathcal{T})$, written as $f^{-1}(U)$ for $U \in \mathcal{T}$. We want to show $V \in \mathcal{T}'$, i.e. $f^{-1}(U) \in \mathcal{T}'$, which is true since $U \in \mathcal{T}$. □