SURGERY OF GALOIS REPRESENTATIONS: MODIFYING THE FROBENIUS ELEMENT

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Proposition 1. *Let K be a local field with finite residue field k, and let $\rho\colon \Gamma_K \to G$ be a group homomorphism. Then, there is a bijection*

$$\left\{\rho'\colon \Gamma_K \to G \;\middle|\; \rho'|_{I_K} = \rho|_{I_K}\right\} \simeq \operatorname{Cent}_G \rho(I_K),$$

given as $\Phi\colon \rho' \mapsto \rho'(\widetilde{\sigma})\rho(\widetilde{\sigma})^{-1}$, where $\widetilde{\sigma}$ is any "Frobenius lift" lifting a topological generator of $\Gamma_K/I_K \simeq \Gamma_k \simeq \widehat{\mathbb{Z}}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_K & \longrightarrow & \Gamma_K & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 1 \\ & & & & \widetilde{\sigma} & \longmapsto & 1 & & \end{array}$$

Proof. First note that I_K and $\widetilde{\sigma}$ topologically generate Γ_K: for any $\tau \in \Gamma_K$, there is a unique profinite integer $\chi \in \widehat{\mathbb{Z}}$ and a unique $\tau' \in I_K$ such that $\tau = \widetilde{\sigma}^{\chi} \cdot \tau'$. Indeed, χ must be equal to $\tau \bmod I_K \in \widehat{\mathbb{Z}}$, and then we must have $\tau' := \widetilde{\sigma}^{-\chi}\tau$.

Injectivity of Φ is clear: if $\Phi(\rho'_1) = \Phi(\rho'_2) = z$ then these (continuous) homomorphisms coincide both on I_K (where they equal $\rho|_{I_K}$) and on $\widetilde{\sigma}$ (where they equal $z\rho(\widetilde{\sigma})$), so they coincide since $\Gamma_K = \overline{\langle \widetilde{\sigma}, I_K\rangle}$.

We now check that Φ is valued in the centralizer of $\rho(I_K)$. Let ρ' be such that $\rho'|_{I_K} = \rho|_{I_K}$ and let $\tau \in I_K$. We must check that

$$\underbrace{\rho(\tau)}_{=\rho'(\tau)}\, \rho'(\widetilde{\sigma})\rho(\widetilde{\sigma})^{-1} = \rho'(\widetilde{\sigma})\rho(\widetilde{\sigma})^{-1}\rho(\tau),$$

i.e., that $\rho'(\widetilde{\sigma}^{-1}\tau\widetilde{\sigma}) = \rho(\widetilde{\sigma}^{-1}\tau\widetilde{\sigma})$, but this follows from $\rho'|_{I_K} = \rho|_{I_K}$ as $I_K \triangleleft \Gamma_K$.

Finally, we prove surjectivity. Let $z \in \operatorname{Cent}_G \rho(I_K)$. We define a map ρ' as follows: for each $\tau \in \Gamma_K$, written (uniquely) as $\widetilde{\sigma}^{\chi} \cdot \tau'$ for some $\chi \in \widehat{\mathbb{Z}}$ and $\tau' \in I_K$, we let $\rho'(\tau) = (z\rho(\widetilde{\sigma}))^{\chi}\rho(\tau')$. This map is well-defined, continuous, satisfies $\rho'(\widetilde{\sigma})\rho(\widetilde{\sigma})^{-1} = z$, and $\rho'|_{I_K} = \rho|_{I_K}$. It remains to check that ρ' is a group homomorphism. Consider two elements $\tau_1 = \widetilde{\sigma}^{\chi_1} \cdot \tau'_1$ and $\tau_2 = \widetilde{\sigma}^{\chi_2} \cdot \tau'_2$ of Γ_K. We have $\tau_1\tau_2 = \widetilde{\sigma}^{\chi_1+\chi_2} \cdot (\widetilde{\sigma}^{-\chi_2} \cdot \tau'_1 \cdot \widetilde{\sigma}^{\chi_2})\tau'_2$, so

$$\rho'(\tau_1\tau_2) = (z\rho(\widetilde{\sigma}))^{\chi_1+\chi_2}\rho(\widetilde{\sigma}^{-\chi_2} \cdot \tau'_1 \cdot \widetilde{\sigma}^{\chi_2})\rho(\tau'_2) \quad = (z\rho(\widetilde{\sigma}))^{\chi_1}(z\rho(\widetilde{\sigma}))^{\chi_2}\rho(\widetilde{\sigma}^{-\chi_2} \cdot \tau'_1 \cdot \widetilde{\sigma}^{\chi_2})\rho(\tau'_2)$$

and

$$\rho'(\tau_1)\rho'(\tau_2) = (z\rho(\widetilde{\sigma}))^{\chi_1}\rho(\tau'_1)(z\rho(\widetilde{\sigma}))^{\chi_2}\rho(\tau'_2)$$

so we need to verify whether

$$(z\rho(\widetilde{\sigma}))^{-\chi_2}\rho(\tau'_1)(z\rho(\widetilde{\sigma}))^{\chi_2} \stackrel{?}{=} \rho(\widetilde{\sigma})^{-\chi_2} \cdot \rho(\tau'_1) \cdot \rho(\widetilde{\sigma})^{\chi_2},$$

which reduces (by continuity and an obvious induction) to checking that $z\rho(\widetilde{\sigma})$ and $\rho(\widetilde{\sigma})$ act identically on $\rho(I_K)$ by conjugation, i.e., the case $\chi_2 = 1$. But that case is clear:

$$(z\rho(\widetilde{\sigma}))^{-1}\rho(\tau'_1)(z\rho(\widetilde{\sigma})) = \rho(\widetilde{\sigma})^{-1}z^{-1}\underbrace{\rho(\tau'_1)z}_{\text{commute}}\rho(\widetilde{\sigma}) = \rho(\widetilde{\sigma})^{-1}\rho(\tau'_1)\rho(\widetilde{\sigma}). \qquad \square$$

Proposition 2. *Let $\iota\colon I_K \to G$. Then*

$$\exists \rho \in \operatorname{Hom}(\Gamma_K, G),\, \iota = \rho|_{I_K} \iff \exists g \in G, \forall \tau \in I_K,\, \iota(\widetilde{\sigma}\tau\widetilde{\sigma}^{-1}) = g\iota(\tau)g^{-1}$$

Proof. For ($\Rightarrow$), take $g = \rho(\widetilde{\sigma})$ and compute $\rho(\widetilde{\sigma}\tau\widetilde{\sigma}^{-1})$ in two different ways.

For ($\Leftarrow$), reason as in the previous proof: for $\chi \in \widehat{\mathbb{Z}}$ and $\tau \in I_K$, define $\rho(\widetilde{\sigma}^{\chi} \cdot \tau) := g^{\chi} \cdot \iota(\tau)$, and check that this defines a homomorphism (we have $g^{\chi} \cdot \iota(\tau) = \iota(\widetilde{\sigma}^{\chi}\tau\widetilde{\sigma}^{-\chi}) \cdot g^{\chi}$). $\square$