# Maximally Even Sequences

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**Abstract**

*Maximally even sequences* are maps $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following property, which abstracts a property satisfied by the major scale (in music): for any $k \in \mathbb{Z}$, there is an integer $c_k$ such that all differences $f(n+k) - f(n)$ belong to $\{c_k, c_k + 1\}$ for any $n$. In this article, we give a complete classification of all maximally even sequences.

## 1 Introduction

### 1.1 Context

In music, the major scale is a certain subset of the set of all notes (which, in 12-tone equal temperament, can be identified with $\mathbb{Z}$). This subset has an interesting property: if one knows how many notes of the major scale are between two given notes (the "generic interval"), the number of semitones between these notes (the "specific interval") can only take at most two different (successive) values. This fact is reflected in terminology: when musicians name the interval between two notes of the major scale, they say how far apart in the scale the notes are (a third, a fifth, a sixth, etc.), and add an adjective indicating if the interval is the smaller or the bigger specific interval (for seconds, thirds, sixths and sevenths, we use the adjectives *minor* and *major*; fourths are either *perfect* or *augmented*; fifths are either *diminished* or *perfect*; unisons and octaves are always perfect).

In this article, we study sequences satisfying a condition which abstracts that property:

**Definition 1.1.** A map $f : \mathbb{Z} \to \mathbb{Z}$ is *maximally even* if, for each integer $k \in \mathbb{Z}$, there exists an integer $c_k(f) \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$, the difference $f(n+k) - f(n)$ belongs to $\{c_k(f), c_k(f)+1\}$. Equivalently, $f$ is maximally even if:

$$\forall n, m, k \in \mathbb{Z}, \left| \left| f(n+k) - f(n) \right| - \left| f(m+k) - f(m) \right| \right| \leq 1.$$

The set of maximally even sequences is preserved by horizontal and vertical shifts, by addition of a linear function, and by horizontal and vertical reflections.

Maximal evenness was considered in [Nor89; DK07; DK08] at a lower level of generality — mostly restricted to pseudo-periodic sequences (cf. Definition 1.7). Maximally even sequences are also related to a construction of real numbers known as *Eudoxus real numbers*, exposed in [Art04]: in that context, Example 1.3 and Lemma 2.1 are familiar results.

**Example 1.2.** An example of a maximally even sequence is the major scale, which is the following map extended by $f(n+7k) = f(n) + 12k$:

| $\boldsymbol{n}$ | … | −1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{f(n)}$ | … | −1 | 0 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | … |

**Example 1.3.** For all real numbers $\alpha$, $\beta$, the sequence $n \mapsto \lfloor \alpha n + \beta \rfloor$ is maximally even.

**Example 1.4.** Let $a, b, c \in \mathbb{Z}$ and $\varepsilon \in \{-1, 1\}$. Then, the following sequence is maximally even:

$$f : n \mapsto \begin{cases} an + b + \varepsilon & \text{if } x \geq c \\ an + b & \text{if } x < c. \end{cases}$$

Note that this function is not of the type of Example 1.3.

## 1.2 Main theorem

The main result of this article is Theorem 1.5, which is a classification of maximally even sequences. Roughly speaking, this result says that, for any maximally even sequence $f$, there exists $\alpha \in \mathbb{R}$ such that $f$ is (except perhaps at one single point) obtained by gluing two maps of the form $n \mapsto \lfloor \alpha n + \beta \rfloor$ with $\beta \in \mathbb{R}$. Here is the precise statement:

**Theorem 1.5.** *Let $f$ be a maximally even sequence. Then, the following limit exists:*

$$\alpha = \lim_{n \to \pm\infty} \frac{f(n)}{n}.$$

*Moreover:*

- *if $\alpha$ is irrational, there are two possible situations:*
  - *$f$ is of the following form, for some uniquely defined $\beta \in \mathbb{R}$:*

$$n \mapsto \lfloor \alpha n + \beta \rfloor.$$

  - *$f$ is of the following form, for uniquely defined integers $c, d \in \mathbb{Z}$:*

$$n \mapsto \begin{cases} \lfloor \alpha(n-c) + d \rfloor & \textit{if } n \neq c \\ d-1 & \textit{if } n = c. \end{cases}$$

- *if $\alpha = \frac{p}{q}$ with $p, q$ coprime, there are two possible situations:*
  - *$f$ is of the following form, for some unique integer $c \in \mathbb{Z}$:*

$$n \mapsto \left\lfloor \frac{pn+c}{q} \right\rfloor.$$

  - *either $f$ or $-f$ is of the following form, for some unique pair of integers $(c, \mu)$ which satisfies $p\mu \equiv -c \pmod q$:*

$$n \mapsto \begin{cases} \left\lfloor \frac{pn+c-1}{q} \right\rfloor & \textit{if } n \leq \mu + q - 1 \\ \\ \left\lfloor \frac{pn+c}{q} \right\rfloor & \textit{if } n \geq \mu + 1. \end{cases}$$

## 1.3 Outline

In Section 2, we define candidates for the constants $\alpha$ and $\beta$ such that an arbitrary maximally even sequence is "close to" $n \mapsto \lfloor \alpha n + \beta \rfloor$ (Lemmas 2.1, 2.3 and 2.5).

In Section 3, we classify all the maximally even sequences for which the number $\alpha$ defined in Lemma 2.1 is irrational (Proposition 3.1).

In Section 4, we classify the pseudo-periodic maximally even sequences (Lemmas 4.1 and 4.2).

In Section 5, we classify the remaining maximally even sequences, i.e., those for which $\alpha$ is rational but which are not pseudo-periodic (Proposition 5.4).

Together, these results imply Theorem 1.5.

### 1.4 Notation and terminology

**Definition 1.6.** Let $f : \mathbb{Z} \to \mathbb{Z}$ be a map and $k \in \mathbb{Z}$. We let

$$c_k(f) := \inf_{n\in\mathbb{Z}} \big(f(n+k) - f(n)\big).$$

Using this definition, a sequence $f$ is maximally even if and only if

$$\forall k \in \mathbb{Z}, \quad \forall n \in \mathbb{Z}, \quad f(n) + c_k(f) \leq f(n+k) \leq f(n) + c_k(f) + 1.$$

(This inequality implies that $c_k(f)$ is finite for all $k \in \mathbb{Z}$.)

**Definition 1.7.** Let $q \in \mathbb{N}$ and $p \in \mathbb{Z}$. A function $f : \mathbb{Z} \to \mathbb{Z}$ is *(p,q)-pseudo-periodic* if $f(n+q) = f(n) + p$ for all $n \in \mathbb{Z}$.

For example, the "major scale" of Example 1.2 is a $(12, 7)$-pseudo-periodic maximally even sequence, for which the values of $c_k$ for $k \in \{1, \dots, 7\}$ are given by $\{1, 3, 5, 6, 8, 10, 12\}$.

**Definition 1.8.** Let $k \in \mathbb{N}$. A *k-step* is an ordered pair $(n, m) \in \mathbb{Z}^2$ such that $m = n+k$. The 1-steps are simply called *steps*. The *distance* between two $k$-steps $(n, m)$ and $(n', m')$ is the integer $n' - n'$, or equivalently $m' - m'$.

**Definition 1.9.** The $k$-step $(n, n + k)$ *contains* the $l$-step $(m, m + l)$ if $n \leq m \leq m + l \leq n + k$.

**Definition 1.10.** Let $f$ be a function $\mathbb{Z} \to \mathbb{Z}$ and $k \in \mathbb{N}$:

- a *minor* $k$-step is a $k$-step $(n, m)$ such that $f(m) - f(n) = c_k(f)$;
- a *major* $k$-step is a $k$-step $(n, m)$ such that $f(m) - f(n) = c_k(f) + 1$.

A function is maximally even if and only if every $k$-step is either minor or major, for all $k \geq 1$.

## 2 Defining candidates for $\alpha$ and $\beta$

Let $f$ be a maximally even sequence.

**Lemma 2.1.** *The limit*

$$\alpha(f) = \lim_{n\to\pm\infty} \frac{f(n)}{n}$$

*exists and satisfies, for every $k \in \mathbb{Z}$:*

$$c_k(f) \leq \alpha(f)k \leq c_k(f) + 1. \tag{1}$$

*Proof.* Let $g = f - f(0) + 1$. Then:

$$\begin{aligned} g(a+b) = f(a+b) - f(0) + 1 &\leq f(a) + c_b(f) + 1 - f(0) + 1 \\ &\leq f(a) + [f(b) - f(0)] + 1 - f(0) + 1 \\ &= f(a) + f(b) - 2f(0) + 2 = g(a) + g(b), \end{aligned}$$

so $g$ is a subadditive sequence. Thus, there is a constant $\alpha(f)$ such that:

$$\frac{f(n)}{n} = \frac{g(n)}{n} + o(1) = \alpha(f) + o(1).$$

Now take the specific subsequence $f(kn)$. We have:

$$f(0) + nc_k(f) \leq f(kn) \leq f(0) + n(c_k(f) + 1).$$

Dividing by $n$ and taking the limit as $n \to +\infty$, we obtain Equation (1). In particular:

$$\alpha(f) = \lim_{k\to+\infty} \frac{c_k(f)}{k}.$$

Now, consider the maximally even sequence $\tilde{f} : n \mapsto -f(-n)$. Note that $c_k(\tilde{f})$ is the maximal value of $f(n+k) - f(n)$, hence is either $c_k(f)$ or $c_k(f)+1$. Since $c_k(\tilde{f}) = c_k(f) + O(1)$, we have

$$\alpha(f) = \lim_{k\to+\infty} \frac{c_k(f)}{k} = \lim_{k\to+\infty} \frac{c_k(\tilde{f})}{k} = \alpha(\tilde{f}) = \lim_{k\to+\infty} \frac{\tilde{f}(k)}{k} = \lim_{k\to-\infty} \frac{-f(k)}{-k} = \lim_{k\to-\infty} \frac{f(k)}{k}. \qquad \square$$

**Definition 2.2.** A maximally even sequence $f$ is *rational* if $\alpha(f)$ is rational, and *irrational* otherwise.

**Lemma 2.3.** *Let $f$ be a maximally even sequence. Let:*

$$\beta(f) = \sup_{n\in\mathbb{N}} \big(f(n) - \alpha(f)n\big).$$

*Then $\beta(f)$ is finite and we have, for all $n \in \mathbb{Z}$:*

$$f(n) \leq \alpha(f)n + \beta(f) \leq f(n) + 1.$$

*Proof.* Let $n, n' \in \mathbb{Z}$ be arbitrary integers. Since $f$ is maximally even, we have:

$$f(n) - f(n') \leq c_{n-n'}(f) + 1,$$

which by Equation (1) implies the following inequality:

$$f(n) - f(n') \leq \alpha(f)(n - n') + 1.$$

Rewrite this inequality under the equivalent form:

$$f(n) - \alpha(f)n \leq f(n') - \alpha(f)n' + 1.$$

This equality implies that, as $n$ varies, the expression $f(n) - \alpha(f)n$ takes values in an interval of length at most 1; in particular, its supremum $\beta(f)$ is finite and we have, for all $n \in \mathbb{Z}$:

$$\beta(f) - 1 \leq f(n) - \alpha(f)n \leq \beta(f). \qquad \square$$

A direct (and far-reaching) consequence is the following:

**Corollary 2.4.** *Let $f$ be a maximally even sequence. If an integer $n \in \mathbb{Z}$ satisfies $f(n) \neq \lfloor \alpha(f)n + \beta(f) \rfloor$, then $f(n) = \alpha(f)n + \beta(f) - 1$. In particular, $\alpha(f)n + \beta(f)$ is an integer.*

When $\alpha$ is rational, Lemma 2.3 can be refined: we can replace $\beta$ by a rational number with the same denominator as $\alpha$. This is the content of the following lemma, used in Sections 4 and 5:

**Lemma 2.5.** *Let $f$ be a rational maximally even sequence, with $\alpha(f) = \frac{p}{q}$. Then:*

- *There exists an integer $c \in \mathbb{Z}$ such that, for all $n \in \mathbb{Z}$:*

$$f(n) \leq \frac{pn + c}{q} \leq f(n) + 1.$$

- *Assume that the following condition is satisfied for every $n, n' \in \mathbb{Z}$:*

$$q(f(n) - f(n') - 1) \neq p(n - n').$$

*Then, $c$ can be chosen such that, for all $n \in \mathbb{Z}$:*

$$f(n) = \left\lfloor \frac{pn + c}{q} \right\rfloor.$$

*Proof.* The condition that the integer $c$ must satisfy is:

$$\forall n \in \mathbb{Z}, \qquad qf(n) - pn \leq c \leq qf(n) + q - pn.$$

The existence of such a $c$ amounts to the following inequality, for all $n, n' \in \mathbb{Z}$:

$$qf(n) - pn \leq qf(n') + q - pn'.$$

But maximal evenness implies the equivalent inequality:

$$q(f(n) - f(n')) \leq q\,(c_{n-n'} + 1) \leq p(n - n') + q.$$

The equality $f(n) = \left\lfloor \frac{pn+c}{q} \right\rfloor$ happens whenever there is a strict inequality in the previous inequalities, i.e., when $q(f(n) - f(n')) < p(n - n') + q$. $\square$

## 3 Irrational maximally even sequences

Let $f$ be a maximally even sequence such that the constant $\alpha \coloneqq \alpha(f)$ from Lemma 2.1 is irrational. In this section, we prove the following proposition, which is the first case of Theorem 1.5:

**Proposition 3.1.** *The function $f$ is either equal to*

$$n \mapsto \lfloor \alpha n + \beta \rfloor \qquad\qquad \text{for some (unique) } \beta \in \mathbb{R},$$

*or it is equal to*

$$n \mapsto \begin{cases} \lfloor \alpha(n - c) + d \rfloor & \textit{if } n \neq c \\ d - 1 & \textit{if } n = c \end{cases} \qquad\qquad \text{for some (unique) integers } c, d.$$

*Proof.* Let $\beta(f)$ be as in Lemma 2.3. Assume that there are two values $c_1 \neq c_2$ such that $f(c_i) \neq \lfloor \alpha c_i + \beta(f) \rfloor$. By Corollary 2.4, we have $f(c_i) = \alpha c_i + \beta(f) - 1$ for $i \in \{1, 2\}$. Then:

$$\alpha = \frac{f(c_1) - f(c_2)}{c_1 - c_2} \in \mathbb{Q}$$

which is a contradiction. Therefore, either $f(n) = \lfloor \alpha n + \beta(f) \rfloor$ for all $n$, or there is a single integer $c$ for which this does not hold, and for that value we have $f(c) = \alpha c + \beta(f) - 1$. In the latter case, letting $d \coloneqq f(c) + 1 = \alpha c + \beta(f)$ yields the statement given above.

The only thing left to prove is the uniqueness of $\beta$. Assume that there are two real numbers $\beta$ and $\beta'$ such that, for all values of $n \in \mathbb{Z}$ (except perhaps finitely many), we have:

$$\lfloor \alpha n + \beta \rfloor = \lfloor \alpha n + \beta' \rfloor.$$

Since $\alpha$ is irrational, the sequence $(\alpha n - \lfloor \alpha n \rfloor)_n$ is dense in $(0, 1)$. Thus, the (right-continuous) functions $x \mapsto \lfloor x + \beta \rfloor$ and $x \mapsto \lfloor x + \beta' \rfloor$ coincide on a dense subset of $(0, 1)$, which implies $\beta = \beta'$. $\square$

## 4 Pseudo-periodic maximally even sequences

In this section, we classify maximally even sequence which are $(p, q)$-pseudo-periodic (see Definition 1.7). The main result is the combination of Lemmas 4.1 and 4.2, which together form the first subcase of the rational case of Theorem 1.5. We first show that we can assume $p$ and $q$ coprime:

**Lemma 4.1.** *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a $(p, q)$-pseudo-periodic maximally even sequence. Assume that $p$ and $q$ have a common divisor $d$, and let $p = dp'$ and $q = dq'$. Then $f$ is $(p', q')$-pseudo-periodic.*

*Proof.* We have:

$$\begin{aligned}
\sum_{i=0}^{q-1}\Big[f(i+q')-f(i)-p'\Big] &= \left(\sum_{i=0}^{q'-1}\sum_{j=0}^{d-1}\Big[f\big((i+q'j)+q'\big)-f(i+q'j)\Big]\right)-p'q \\
&= \left(\sum_{i=0}^{q'-1}\sum_{j=0}^{d-1}\Big[f\big(i+q'(j+1)\big)-f(i+q'j)\Big]\right)-p'q \\
&= \left(\sum_{i=0}^{q'-1}\Big[f(q'd+i)-f(i)\Big]\right)-p'q \\
&= \left(\sum_{i=0}^{q'-1}\Big[f(q+i)-f(i)\Big]\right)-p'q \\
&= \left(\sum_{i=0}^{q'-1}p\right)-p'q = q'p-p'q = d(p'q'-p'q') = 0
\end{aligned}$$

Since $(f(i+q')-f(i)-p')_{0\le i<q}$ is a list of integers that sums to 0, it is either identically 0 (i.e., $f$ is $(p',q')$-pseudo-periodic), or it takes values both $\geq 1$ and $\leq -1$, but the latter option is impossible as $f$ is maximally even. Therefore, $f$ is $(p',q')$-pseudo-periodic. □

**Lemma 4.2.** *Let $f$ be a $(p,q)$-pseudo-periodic maximally even sequence, with $p$ and $q$ coprime. Then, there is a unique integer $c\in\mathbb{Z}$ such that for all $n\in\mathbb{Z}$:*

$$f(n)=\left\lfloor\frac{pn+c}{q}\right\rfloor.$$

*Proof.* We have:

$$\alpha(f)=\lim_{k\to\pm\infty}\frac{f(kq)}{kq}=\lim_{k\to\pm\infty}\frac{f(0)+kp}{kq}=\frac{p}{q},$$

so we are in the rational case. We want to show that there is a $c\in\mathbb{Z}$ such that:

$$f(n)=\left\lfloor\frac{pn+c}{q}\right\rfloor.$$

By the second point of Lemma 2.5, it is enough to show that for any $n,n'\in\mathbb{Z}$ we have:

$$q(f(n)-f(n')-1)\neq p(n-n').$$

By contradiction, assume that there are integers $n$, $n'$ such that $q(f(n)-f(n')-1)=p(n-n')$. Since $q$ and $p$ are coprime, $q$ divides $n-n'$. Let $n-n'=q\delta$. Since $f$ is $(p,q)$-pseudo-periodic, we have $f(n)=f(n'+q\delta)=f(n')+p\delta$. Therefore:

$$q(f(n)-f(n')-1)=q(p\delta-1)=p(n-n')-q\neq p(n-n')$$

which is a contradiction.

It remains only to prove that $c$ is unique. Let $c<c'$ be two distinct integers. Since $p$ and $q$ are coprime, there is an integer $N$ such that $pN\equiv -c' \pmod q$. Then:

$$\left\lfloor\frac{pN+c}{q}\right\rfloor\leq\frac{pN+c}{q}<\frac{pN+c'}{q}=\left\lfloor\frac{pN+c'}{q}\right\rfloor.$$ □

**Back to music.** The major scale from music (Example 1.2) is a $(12,7)$-pseudo-periodic maximally even sequence, and it is in fact given by:

$$n \mapsto \left\lfloor \frac{12n+5}{7} \right\rfloor.$$

Since 12 and 7 are coprime, there are exactly seven different maximally even 7-note scales: they are the seven diatonic modes of the major scale (obtained by changing "5"). We now list the $(12,q)$-pseudo-periodic maximally even sequences for $q \in \{1,\dots,12\}$. Most of them have musical names.

| $q$ | # of modes | Notes (for one of the representatives) | Description |
|---|---|---|---|
| 1 | 1 | $\{0\}$ | Single note |
| 2 | 1 | $\{0,6\}$ | Tritone |
| 3 | 1 | $\{0,4,8\}$ | Augmented chord |
| 4 | 1 | $\{0,3,6,9\}$ | Diminished seventh chord |
| 5 | 5 | $\{0,2,4,7,9\}$ | Major pentatonic scale |
| 6 | 1 | $\{0,2,4,6,8,10\}$ | Whole-tone scale |
| 7 | 7 | $\{0,2,4,5,7,9,10,11\}$ | Major scale |
| 8 | 2 | $\{0,2,3,5,6,8,9,11\}$ | Octatonic (diminished) scale |
| 9 | 3 | $\{0,1,2,4,5,6,8,9,10\}$ | Messiaen's third mode |
| 10 | 5 | $\{0,1,2,3,4,6,7,8,9,10\}$ | Messiaen's seventh mode |
| 11 | 11 | $\{0,1,2,3,4,5,6,7,8,9,10\}$ | All notes but one |
| 12 | 12 | $\{0,1,2,3,4,5,6,7,8,9,10,11\}$ | Chromatic scale |

## 5 Non-pseudo-periodic rational maximally even sequences

In this section, we classify the non-pseudo-periodic rational maximally even sequences. The main result is Proposition 5.4, which is the second subcase of the rational case of Theorem 1.5.

Let $f$ be a maximally even sequence. We define the lower densities of major and minor $k$-steps:

**Definition 5.1.** Let

$$M_k := \{n \in \mathbb{N} \mid f(n+k) - f(n) = c_k(f) + 1\}.$$

The *lower density of major $k$-steps* is:

$$L_k := \liminf_{n\to\infty} \frac{|\{-n,-n+1,\dots,-1,0,1,\dots,n-2,n-1\} \cap M_k|}{2n},$$

and similarly the *lower density of minor $k$-steps* is:

$$\ell_k := \liminf_{n\to\infty} \frac{|\{-n,-n+1,\dots,-1,0,1,\dots,n-2,n-1\} \cap M_k^c|}{2n}.$$

**Lemma 5.2.** *Let $k \in \mathbb{N}$. Either $f$ has at most $k$ major (resp. minor) $k$-steps, or it has infinitely many, with a strictly positive lower density.*

*Proof.* Assume that $f$ has at least $(k+1)$ major $k$-steps. Then, there are two such $k$-steps $(n, n+k)$ and $(n+\delta, n+\delta+k)$ whose distance $\delta$ is a multiple $lk$ of $k$. Thus:

$$f(n+\delta+k) - f(n) = \sum_{i=0}^{l} \Big[ f(n+(i+1)k) - f(n+ik) \Big] \geq (l+1)c_k(f) + 2.$$

Assume that some $(\delta+k)$-step $(m, m+\delta+k)$ contains only minor $k$-steps. Then:

$$f(m+\delta+k) - f(m) = \sum_{i=0}^{l} \Big[ f(m+(i+1)k) - f(m+ik) \Big] = (l+1)c_k(f),$$

which is impossible since $f$ is maximally even. Therefore, every $(\delta + k)$-step contains at least one major $k$-step. This gives an explicit lower bound on $L_k$:

$$L_k \geq \frac{1}{\delta + k} > 0.$$

The same proof applies to minor $k$-steps. $\square$

We now assume that $\alpha(f) = \frac{p}{q}$, with $p$ and $q$ coprime.

**Lemma 5.3.** *Either $f$ has at most $q$ major $q$-steps, or it has at most $q$ minor $q$-steps.*

*Proof of Lemma 5.3.* We define:

$$u_n := \frac{1}{q} \sum_{i=0}^{q-1} \Big[ f(i + nq) - f(i) \Big] = \frac{1}{q} \sum_{i=0}^{q-1} \sum_{j=0}^{n-1} \Big[ f(i + (j+1)q) - f(i + jq) \Big].$$

By definition of $L_q$, we have:

$$u_n \geq n(c_q(f) + L_q) + o(1).$$

Moreover, by Lemma 2.1:

$$\frac{u_n}{n} = \frac{1}{q} \sum_{i=0}^{q-1} \frac{f(i + nq) - f(i)}{n} = \frac{1}{q} \sum_{i=0}^{q-1} q\alpha(f) + o(1) \to q\alpha(f).$$

And thus:

$$p = q\alpha(f) \geq c_q(f) + L_q.$$

Similarly, we have $u_n \leq qn(c_q(f) + 1 - \ell_q)$ and thus:

$$p = q\alpha(f) \leq c_q(f) + 1 - \ell_q.$$

In particular, the integer $p$ is between $c_q(f)$ and $c_q(f) + 1$, and therefore must be equal to one of them. This implies that either $L_q = 0$ or $\ell_q = 0$. By Lemma 5.2, we conclude that $f$ either has at most $q$ major $q$-steps (if $L_q = 0$) or at most $q$ minor $q$-steps (if $\ell_q = 0$). $\square$

Finally, we prove the only missing piece of Theorem 1.5:

**Proposition 5.4.** *Either $f$ or $-f$ is of the following form, for a unique pair of integers $(c, \mu)$ satisfying $p\mu \equiv -c \pmod q$:*

$$f(n) = \begin{cases} \left\lfloor \frac{pn+c-1}{q} \right\rfloor & \textit{if } n \leq \mu + q - 1 \\ \\ \left\lfloor \frac{pn+c}{q} \right\rfloor & \textit{if } n \geq \mu + 1. \end{cases}$$

*Proof.* In Lemma 5.3, we have shown that $f$ has either at most $q$ major $q$-steps, or at most $q$ minor $q$-steps. Replacing $f$ by $-f$ if needed, we assume that $f$ has at most $q$ major $q$-steps. In particular, $c_q(f) = p$. By Lemma 2.5, there is an integer $c \in \mathbb{Z}$ such that:

$$\forall n, qf(n) \leq pn + c \leq q(f(n) + 1).$$

Let $M$ be the finite set of major $q$-steps of $f$, i.e.:

$$M = \{n \in \mathbb{Z} \mid f(n + q) = f(n) + p + 1\}.$$

Since $f$ is not $(p,q)$-pseudo-periodic, the set $M$ is not empty. Let $\mu$ be its smallest element and $\nu$ be its largest element.

The function $f$ restricted to $\{n \in \mathbb{Z} \mid n < \mu + q\}$ is $(p,q)$-pseudo-periodic and therefore:

$$f(n) = \left\lfloor \frac{pn+\gamma}{q} \right\rfloor \text{ if } n < \mu + q$$

for some uniquely defined $\gamma \in \mathbb{Z}$ (it is easy to see that Lemma 4.2 holds for functions which are defined only for $n$ small or big enough). There are two possible situations:

- If $qf(n) \leq pn + c < q(f(n)+1)$ for all $n < \mu + q$, then we have $c = \gamma$ by uniqueness of $\gamma$.
- If not, then $pn + c = q(f(n)+1)$ for some $n < \mu + q$ and then we must have $\gamma = c - 1$ by existence and uniqueness of $\gamma$.

The same reasoning applies for integers larger than $\nu$. Hence, for some uniquely defined $\gamma' \in \{c-1, c\}$:

$$f(n) = \left\lfloor \frac{pn+\gamma'}{q} \right\rfloor \text{ if } n > \nu.$$

Let $n \geq \nu + q$, and let $e$ be such that $n - eq < \mu$. Then:

$$\begin{aligned}
\left|M \cap \{n + q\mathbb{Z}\}\right| &= \left|M \cap \{n-q, n-2q, \ldots, n-eq\}\right| \\
&= f(n) - f(n-eq) - ep \\
&= \left\lfloor \frac{pn+\gamma'}{q} \right\rfloor - \left\lfloor \frac{p(n-eq)+\gamma}{q} \right\rfloor - ep \\
&= \left\lfloor \frac{pn+\gamma'}{q} \right\rfloor - \left\lfloor \frac{pn+\gamma}{q} \right\rfloor .
\end{aligned}$$

Since $\gamma, \gamma' \in \{c-1, c\}$, this can only be nonzero if $pn \equiv -c \pmod q$. Since $p$ and $q$ are coprime, exactly one residue class of $n$ mod $q$ satisfies this, so we now assume that $n$ belongs to that class, so $M = M \cap \{n + q\mathbb{Z}\}$. Since $|M \cap \{n + q\mathbb{Z}\}| = |M| \geq 1$, we have $\gamma' > \gamma$, so $\gamma = c - 1$ and $\gamma' = c$, and then $|M| = |M \cap \{n + q\mathbb{Z}\}| = 1$. We have shown that $M$ is a singleton, containing a single element $\mu = \nu$ which satisfies $p\mu \equiv -c \pmod q$. This gives a complete description of $f$:

$$f(n) = \begin{cases} \left\lfloor \frac{pn+c-1}{q} \right\rfloor & \text{if } n < \mu + q \\ \\ \left\lfloor \frac{pn+c}{q} \right\rfloor & \text{if } n > \mu \end{cases} .$$

(The two definitions match where they overlap because $p\mu \equiv -c \pmod q$.) □

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