NILPOTENT ARTIN–SCHREIER THEORY

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A prime number p is fixed. If F is a field, we denote by $\Gamma_F := \mathrm{Gal}(F^{\mathrm{sep}}|F)$ its absolute Galois group.

1. Parametrization of extensions in characteristic p

We fix a group G and a field F of characteristic p.

1.1. Extensions and cohomology classes.

Definition 1.1. *A G-*extension *of F is an étale F-algebra K together with an action of G, such that there is a G-equivariant F^{sep}-algebra isomorphism between $K \otimes_F F^{\mathrm{sep}}$ and the ring of maps $f : G \to F^{\mathrm{sep}}$ (on which G acts via $(g.f)(h) = f(hg)$). An* isomorphism *between two G-extensions of F is a G-equivariant isomorphism between the corresponding étale F-algebras.*

We denote the set of isomorphism classes of G-extensions of F by $\mathrm{\acute{E}tExt}(G, F)$, often confusing an element of $\mathrm{\acute{E}tExt}(G, F)$ with a representative K of the corresponding isomorphism class, and denoting by $\mathrm{Aut}(K)$ the group of its G-equivariant F-algebra automorphisms.

The set $\mathrm{\acute{E}tExt}(G, F)$ is in natural bijection with the cohomology set $H^1(\Gamma_F, G)$, i.e., the set of conjugacy classes of continuous group homomorphisms $\gamma : \Gamma_F \to G$.[1] The stabilizer $\mathrm{Stab}_G(\gamma)$ of such a homomorphism $\gamma : \Gamma_F \to G$ with respect to the conjugation action of G is the centralizer of the image of γ in G. Moreover, if an extension $K \in \mathrm{\acute{E}tExt}(G, F)$ corresponds to $[\gamma] \in H^1(\Gamma_F, G)$, then $\mathrm{Aut}(K) \cong \mathrm{Stab}_G(\gamma)$, and K is a field if and only if γ is surjective.

1.2. The general parametrization principle.

We describe a principle for parametrizing G-extensions of F, following the general method used in [WY92] and [BG14, Proposition 1]. For this, we make the following definition:

Definition 1.2. *An F-*geometrization *of G is a group $G_{F^{\mathrm{sep}}}$ equipped with an action of Γ_F and a Γ_F-equivariant group homomorphism $\sigma : G_{F^{\mathrm{sep}}} \to G_{F^{\mathrm{sep}}}$, such that:*

- *The subgroup of σ-invariant elements of $G_{F^{\mathrm{sep}}}$ is exactly G;*
- *The subgroup of Γ_F-invariant elements of $G_{F^{\mathrm{sep}}}$, which we denote by G_F, contains G.*

The multiplicative Artin–Schreier map *of $G_{F^{\mathrm{sep}}}$ is the Γ_F-equivariant map $\wp : G_{F^{\mathrm{sep}}} \to G_{F^{\mathrm{sep}}}$ defined by $g \mapsto \sigma(g)g^{-1}$.*

Example 1.3. Let $\mathcal{G}$ be an algebraic group over $\mathbb{F}_p$. The group $\mathcal{G}(F^{\mathrm{sep}})$, equipped with its natural Γ_F-action and Frobenius σ, is an F-geometrization of $\mathcal{G}(\mathbb{F}_p)$. This is the situation the definition is trying to generalize.

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[1] If $\varphi : \Gamma_F \to G$ is a morphism, let K be the fixed sub-F-algebra of $\mathrm{Maps}(G, F^{\mathrm{sep}})$ under the action of Γ_F defined by $(\sigma.f)(h) = \sigma(f(\varphi(\sigma)^{-1}h))$. Let $f \in K$. The relation $f(\varphi(\sigma)^{-1}h) = \sigma^{-1}(f(h))$ implies that the value of f on each orbit under left multiplication by $\mathrm{Im}(\varphi)$ is determined by a single element of F^{sep}. For $\sigma \in \ker(\varphi)$, the relation $f(h) = \sigma^{-1}(f(h))$ implies that f takes values in the fixed subfield $F' := (F^{\mathrm{sep}})^{\ker(\varphi)}$. Hence the algebra K is isomorphic to the G-extension $\mathrm{Maps}(\mathrm{Im}(\varphi)\backslash G, F')$.

Conversely, if K is a G-extension, consider a field F' which is a factor of K and let H be the subgroup of Γ_F fixing it. We can show (using the fact that $K \otimes F^{\mathrm{sep}} \simeq \mathrm{Maps}(G, F^{\mathrm{sep}})$) that F' is a Galois extension of F, whose Galois group is the subgroup G' of G preserving it, so that we have an isomorphism $\Gamma_F/H \simeq G'$, inducing a homomorphism $\Gamma_F \to G$ (of kernel H and image G'). Choosing another factor gives a conjugate homomorphism, as G acts transitively on the set of factors of K (because it acts transitively on the set of factors of $K \otimes F^{\mathrm{sep}} \simeq (F^{\mathrm{sep}})^{|G|}$, and each factor of K corresponds to at least one of these factors).

We fix an F-geometrization $G_{F^{\mathrm{sep}}}$ of G. Note that, as the actions of σ and Γ_F commute, we have $\sigma(G_F) \subseteq G_F$. We define an action of $G_{F^{\mathrm{sep}}}$ on itself by the formula:

$$g.m = \sigma(g)mg^{-1}.$$

Note that $g.1 = \wp(g)$. This action restricts to an action of G_F on itself, whose set of orbits we denote by $G_F /\!\!/_{G_F}$. We now prove:

Proposition 1.4. *There is a bijection between the set of G_F-orbits of elements of $G_F \cap \wp(G_{F^{\mathrm{sep}}})$ and the kernel of the map of pointed sets $H^1(\Gamma_F, G) \to H^1(\Gamma_F, G_{F^{\mathrm{sep}}})$ (in non-abelian group cohomology).*

Proof. Let $m \in G_F \cap \wp(G_{F^{\mathrm{sep}}})$. We fix a $g \in G_{F^{\mathrm{sep}}}$ such that $\wp(g) = m$. We define a 1-coboundary $\gamma_g : \Gamma_F \to G_{F^{\mathrm{sep}}}$ by the formula $\gamma_g(\tau) = g^{-1}\tau(g)$. We show that γ_g is valued in G, i.e., that $\gamma_g(\tau)$ is σ-invariant for all $\tau \in \Gamma_F$:

$$\begin{aligned}\sigma(\gamma_g(\tau)) &= \sigma(g^{-1}\tau(g)) \\ &= g^{-1}m^{-1}\tau(\underbrace{m}_{\in G_F} g) \\ &= g^{-1}\tau(g).\end{aligned}$$

Therefore, γ_g defines a 1-coboundary $\Gamma_F \to G$, i.e., a group homomorphism (the action is trivial). Note that its image in $H^1(\Gamma_F, G_{F^{\mathrm{sep}}})$ is trivial by definition. If $\bar{g} \in G_{F^{\mathrm{sep}}}$ was a different element such that $\wp(\bar{g}) = m$, then the equality $\wp(g) = \wp(\bar{g})$ rewrites as $\sigma(\bar{g}^{-1}g) = \bar{g}^{-1}g$, i.e., there is a $\delta \in G$ such that $g = \bar{g}\delta$. But then:

$$\gamma_{\bar{g}}(\tau) = \bar{g}^{-1}\tau(\bar{g}) = \delta^{-1}g^{-1}\tau(g)\delta = \delta^{-1}\gamma_g(\tau)\delta,$$

showing that γ_g and $\gamma_{\bar{g}}$ are conjugate group homomorphisms $\Gamma_F \to G$, and thus define the same element in $\ker(H^1(\Gamma_F, G) \to H^1(\Gamma_F, G_{F^{\mathrm{sep}}}))$, which therefore only depends on m. Now assume that $m' \in G_F \cap \wp(G_{F^{\mathrm{sep}}})$ is in the same G_F-orbit as m, for example $m' = \mu.m$ for some $\mu \in G_F$. Let $g' = \mu g$, and note that $\wp(g') = \mu.\wp(g) = \mu.m = m'$. We have:

$$\gamma_{g'}(\tau) = g^{-1}\mu^{-1}\tau(\underbrace{\mu}_{\in G_F} g) = g^{-1}\tau(g) = \gamma_g(\tau)$$

showing that the element of $\ker(H^1(\Gamma_F, G) \to H^1(\Gamma_F, G_{F^{\mathrm{sep}}}))$ associated to m only depends on its G_F-orbit. We have constructed the desired map. It remains to see that it is a bijection.

Surjectivity. Let $\gamma : \Gamma_F \to G$ be a group morphism whose image in $H^1(\Gamma_F, G_{F^{\mathrm{sep}}})$ is trivial, i.e., there is some $g \in G_{F^{\mathrm{sep}}}$ such that $\gamma(\tau) = g^{-1}\tau(g)$ for all $\tau \in \Gamma_F$. Let $m = \wp(g)$. We show that $m \in G_F$ by showing that it is Γ_F-invariant:

$$\forall \tau \in \Gamma_F, \quad \tau(m) = \tau(\sigma(g)g^{-1}) = \sigma(\tau(g))\tau(g)^{-1} = \sigma(g\underbrace{\gamma(\tau)}_{\in G})\gamma(\tau)^{-1}g^{-1} = \sigma(g)g^{-1} = m.$$

Therefore, m is an element of $G_F \cap \wp(G_{F^{\mathrm{sep}}})$ and, by construction, it is mapped to the cohomology class of γ.

Injectivity. Let m, m' be elements of $G_F \cap \wp(G_{F^{\mathrm{sep}}})$ defining the same cohomology class in $\ker(H^1(\Gamma_F, G) \to H^1(\Gamma_F, G_{F^{\mathrm{sep}}}))$. Pick elements $g, g' \in G_{F^{\mathrm{sep}}}$ such that $m = \wp(g)$ and $m' = \wp(g')$, and define $\gamma := \tau \mapsto g^{-1}\tau(g)$ and $\gamma' := \tau \mapsto (g')^{-1}\tau(g')$. By hypothesis, there is a $\delta \in G$ such that $\gamma' = \delta^{-1}\gamma\delta$. We obtain:

$$\forall \tau \in \Gamma_F, \quad (g')^{-1}\tau(g') = \delta^{-1}g^{-1}\tau(g)\delta$$

which rewrites as:

$$\forall \tau \in \Gamma_F, \quad (g')^{-1}\tau(g') = \delta^{-1}g^{-1}\tau(g)\underbrace{\delta}_{\in G \subseteq G_F}.$$

The element $\mu := g'\delta^{-1}g^{-1}$ is then Γ_F-invariant, and hence belongs to G_F. We have $g' = \mu g\delta$ and therefore:

$$m' = \wp(g') = \sigma(\mu g \underbrace{\delta}_{\in G})\delta^{-1}g^{-1}\mu^{-1} = \mu.\wp(g) = \mu.m$$

showing that m and m' are in the same G_F-orbit. □

Definition 1.5. *We say that* $G_{F^{\mathrm{sep}}}$ satisfies **(Trans)** *if* $G_F \subseteq \wp(G_{F^{\mathrm{sep}}})$. *We say that* $G_{F^{\mathrm{sep}}}$ satisfies **(H90)** *if the map of pointed sets* $H^1(\Gamma_F, G) \to H^1(\Gamma_F, G_{F^{\mathrm{sep}}})$ *is trivial.*

Note that **(Trans)** ensures that the set of G_F-orbits of elements of $G_F \cap \wp(G_{F^{\mathrm{sep}}})$ is simply $G_F /\!\!/_{G_F}$, and that **(H90)** ensures that kernel of the map $H^1(\Gamma_F, G) \to H^1(\Gamma_F, G_{F^{\mathrm{sep}}})$ is all of $H^1(\Gamma_F, G)$. By the remarks of Subsection 1.1, $H^1(\Gamma_F, G)$ is in bijection with the set of isomorphism classes of G-extensions of F. We obtain the following corollary:

Corollary 1.6. *Assume that* $G_{F^{\mathrm{sep}}}$ *satisfies both* **(Trans)** *and* **(H90)**. *Then, there is a bijection*

$$\text{ÉtExt}(G, F) \overset{\sim}{\longleftrightarrow} G_F /\!\!/_{G_F}.$$

Example 1.7. The result of Corollary 1.6 specializes to well-known theories:

- The group $W(F^{\mathrm{sep}})$ of Witt vectors over F^{sep} is an F-geometrization of $\mathbb{Z}_p$: we retrieve Artin–Schreier-Witt theory. The case of Witt vectors of length 1 yields back ordinary Artin–Schreier theory.
- The group $\mathrm{GL}_n(F^{\mathrm{sep}})$ is an F-geometrization of $\mathrm{GL}_n(\mathbb{F}_p)$: we retrieve the theory of étale φ-modules of dimension n (cf. [FO22, Subsection 3.2], and notably Remark 3.24). In particular, the case $n = 1$ gives a special case of Kummer theory, namely the parametrization of $\mathbb{Z}/(q-1)\mathbb{Z}$-extensions.

2. p-groups and Lie algebras

2.1. Definitions.

Definition 2.1. *A* Lie $\mathbb{Z}_p$-algebra *is a* $\mathbb{Z}_p$*-module* L *equipped with a Lie bracket* $[\bullet, \bullet] : L^2 \to L$ *which is* $\mathbb{Z}_p$*-bilinear, alternating, and satisfies the Jacobi identity:*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

We say that L *is* abelian *if its Lie bracket is identically zero. An* ideal *of* L *is a submodule* I *such that* $[L, I] \subseteq I$. *The* center *of* L *is the ideal* $Z(L)$ *formed of elements* x *such that* $[L, x] = 0$.

Let L be a Lie $\mathbb{Z}_p$-algebra. We can form the quotient of L by an ideal I to obtain a Lie algebra L/I. For elements $x_1, \ldots, x_n \in L$, we use the notation

$$[x_1, \ldots, x_n] := \underbrace{[[\cdots[[}_{n-1} x_1, x_2], x_3], \ldots, x_n].$$

We say that L is *nilpotent* if there exists an integer n such that $[x_1, \ldots, x_{n+1}]$ vanishes for all $x_1, \ldots, x_{n+1} \in L$. The smallest such n is then the *nilpotency class* of L. If L has nilpotency class $n \geq 1$, then the quotient $L/Z(L)$ has nilpotency class $n - 1$.

Example 2.2. Only the trivial Lie algebra $L = 0$ has nilpotency class 0. Lie algebras of nilpotency class 1 are abelian Lie algebras. Lie algebras of nilpotency class 2 are nonabelian Lie algebras L for which $L/Z(L)$ is abelian, i.e., $[L, L] \subseteq Z(L)$.

2.2. **The Lazard Correspondence.** Let L be a Lie $\mathbb{Z}_p$-algebra of nilpotency class $< p$. We define a group law $\circ$ on L via the *truncated Baker-Campbell-Hausdorff formula*:

$$x \circ y := x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, y, y] - \frac{1}{12}[x, y, x] + \ldots$$

where the sum is including only the finitely many terms of the Baker-Campbell-Hausdorff formula which do not involve p-th commutators, thus involving only denominators coprime to p. For instance, for groups of nilpotency class ≤ 2 (in odd characteristic), the formula simplifies to $x \circ y = x + y + \frac{1}{2}[x, y]$, and in this case the Lie bracket is determined by the law $\circ$ via $[x, y] = x \circ y \circ (-x) \circ (-y)$. The operation transforming the Lie algebra L into the group $(L, \circ)$ leads to the *Lazard correspondence*, which is an equivalence of categories:

$$\{p\text{-groups of nilpotency class} < p\} \longleftrightarrow \{\text{finite Lie } \mathbb{Z}_p\text{-algebras of nilpotency class} < p\}.$$

This correspondence was introduced by Lazard in [Laz54] (see also [CGV12] or [Abr98, section 1.2]), and it may be seen as analogous to the classical Lie correspondence (between Lie groups and Lie algebras).

Subgroups of $(L, \circ)$ correspond to subalgebras of L, normal subgroups correspond to ideals (the quotients then correspond to each other), and the center of the Lie algebra L is also the center of the group $(L, \circ)$. Note that L is an abelian Lie algebra if and only if $(L, \circ)$ is abelian, in which case the laws $+$ and $\circ$ coincide.

3. Nilpotent Artin–Schreier theory

3.1. Lifts in characteristic zero.

We need an additional tool, which is constructed in [BM90, Subsection 1.1]:

Theorem 3.1. *Let F be a field of characteristic p. There exists a $\mathbb{Z}_p$-algebra $O(F)$, unique up to isomorphism, such that:*

- $O(F)/pO(F) \simeq F$
- *$O(F)$ is p-adically complete, i.e., $O(F) = \varprojlim O(F)/p^n O(F)$*
- *$O(F)$ is flat over $\mathbb{Z}_p$, i.e., $O(F)$ has no non-zero p-torsion elements.*

The ring $O(F)$ can be constructed as a subring of $W(F)$, such that $p^k O(K) = O(K) \cap \mathrm{Ver}^k(W(K))$, where $k \geq 0$ and Ver is the Verschiebung map. If $F'|F$ is a Galois extension of fields of characteristic p, the Galois group $\mathrm{Gal}(F'|F)$ acts naturally on $O(F')$, and the subring of invariant elements is precisely $O(F)$. Moreover, the Frobenius map can be lifted into a $\mathbb{Z}_p$-linear map $\sigma : O(F) \to O(F)$ (reducing to $\sigma : x \mapsto x^p$ modulo p) whose fixed points are exactly the elements of $\mathbb{Z}_p$.

Example 3.2. Let k be a perfect field. Then, $O(k) = W(k)$, $O(k(\!(t)\!)) = W(k)(\!(t)\!)$, and $O(k(t))$ is the p-adic completion of the localization of $W(k)[t]$ at (p).

3.2. Nilpotent Artin–Schreier theory.

Let G be a p-group of nilpotency class $< p$. The Lazard correspondence gives a natural candidate for an F-geometrization of G to which Corollary 1.6 can be applied. Indeed, consider the finite Lie $\mathbb{Z}_p$-algebra L corresponding to G, and define $G_{F^{\mathrm{sep}}} := (L \otimes O(F^{\mathrm{sep}}), \circ)$, equipped with its natural Γ_F-action and Frobenius σ.

Proposition 3.3. *$G_{F^{\mathrm{sep}}}$ is an F-geometrization of G satisfying* **(Trans)** *and* **(H90)***.*

Proof. We have $(G_{F^{\mathrm{sep}}})^{\sigma} = (L \otimes O(\mathbb{F}_p), \circ) = (L, \circ) = G$. Note also that $(G_{F^{\mathrm{sep}}})^{\Gamma_F} = (L \otimes O(F), \circ)$. Instead of proving **(Trans)**, we prove the stronger claim that $\wp(G_{F^{\mathrm{sep}}}) = G_{F^{\mathrm{sep}}}$. Instead of proving **(H90)**, we prove the stronger claim that $H^1(\Gamma_F, G_{F^{\mathrm{sep}}})$ is trivial.

We first deal with the case $L = \mathbb{Z}/p\mathbb{Z}$, in which case $G_{F^{\mathrm{sep}}}$ is simply the additive group F^{sep}. Showing that $\wp$ is surjective requires checking that, for each $x \in F^{\mathrm{sep}}$, there is a $y \in F^{\mathrm{sep}}$ such that $x = \sigma(y) - y$. Since $x = \sigma(y) - y$ is a separable polynomial equation in y, this is clear. The fact that $H^1(\Gamma_F, F^{\mathrm{sep}}) = 1$ follows from [Ser62, Chap. X, §1, Prop. 1].

We now prove both the triviality of $H^1(\Gamma_F, G_{F^{\mathrm{sep}}})$ and the surjectivity of $\wp$ by induction on the size of L. Assume that L is nonzero, and choose a subalgebra I in the center of L, isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Let $Q = L/I$. We have the exact sequence:

$$0 \to I \to L \to Q \to 0$$

by flatness of $O(F^{\mathrm{sep}})$ and properties of the Lazard correspondence, it induces an exact sequence (of groups equipped with a Γ_F-action):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (I \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}}), \circ) & \longrightarrow & (L \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}}), \circ) & \longrightarrow & (Q \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}}), \circ) & \longrightarrow & 1 \\ & & \wr\!| & & & & & & \\ & & F^{\mathrm{sep}} & & & & & & \end{array}$$

We obtain the following exact sequence in non-abelian Galois cohomology:

$$H^1(\Gamma_F, F^{\mathrm{sep}}) \to H^1(\Gamma_F, (L \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}}), \circ)) \to H^1(\Gamma_F, (Q \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}}), \circ)).$$

We have $H^1(\Gamma_F, F^{\mathrm{sep}}) = 1$ by the case $L = \mathbb{Z}/p\mathbb{Z}$ and $H^1(\Gamma_F, (Q \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}}), \circ)) = 1$ by induction hypothesis, so that $H^1(\Gamma_F, (L \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}}), \circ)) = 1$.

We now show the surjectivity of $\wp$. Let $x \in L \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}})$, and let $\bar{x}$ be its projection in $Q \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}})$. By induction hypothesis, there is a $\bar{y} \in Q \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}})$ such that $\wp(\bar{y}) = \bar{x}$. Choose an arbitrary lifting $y_0 \in L \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}})$ of $\bar{y}$. Then $x - \wp(y_0)$ belongs to $I \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}})$ and hence, by the case $L = \mathbb{Z}/p\mathbb{Z}$, there is a $z \in I \otimes_{\mathbb{Z}_p} O(F^{\mathrm{sep}})$ such that $x = \wp(y_0) + \wp(z)$. As z is central, we have:

$$x = \sigma(y_0) \circ (-y_0) \circ \sigma(z) \circ (-z) = \sigma(y_0) \circ \sigma(z) \circ (-y_0) \circ \circ(-z) = \sigma(y_0 + z) \circ (-y_0 - z) = \wp(y_0 + z)$$

establishing the surjectivity of $\wp$. ☐

We define a left action of $(L \otimes O(F), \circ)$ on $L \otimes O(F)$ by:

$$g.m := \sigma(g) \circ m \circ (-g).$$

We write $L \otimes O(F)/\!\!/_{O(F)}$ for the set of $(L \otimes O(F), \circ)$-orbits of $L \otimes O(F)$.

Theorem 3.4 (Parametrization)**.** *There is a bijection between the sets* ÉtExt(G, F) *and* $L \otimes O(F)/\!\!/_{O(F)}$. *Moreover, if $K|F$ is a G-extension of F and if $m \in L \otimes O(F)$ is an element of the orbit corresponding to K, then* $\mathrm{Stab}_{(L \otimes O(F), \circ)}(m) \cong \mathrm{Aut}(K)$.

Proof. The first part follows from Corollary 1.6 and Proposition 3.3. For the second part, let $g \in L \otimes O(F^{\mathrm{sep}})$ be such that $m = g.0$, so that a homomorphism $\gamma : \Gamma_F \to (L, \circ)$ associated to K is given by $\tau \mapsto (-g) \circ \tau(g)$. As we remarked in Subsection 1.1, we have an isomorphism $\mathrm{Aut}(K) \simeq \mathrm{Stab}_{(L,\circ)}(\gamma)$. For all $h \in L$, we have:

$$\begin{aligned} h \in \mathrm{Stab}_{(L,\circ)}(\gamma) &\iff \forall \tau \in \Gamma_F,\ h \circ (-g) \circ \tau(g) \circ \underbrace{(-h)}_{=\tau(-h)} = (-g) \circ \tau(g) \\ &\iff \forall \tau \in \Gamma_F,\ \tau(g \circ (-h) \circ (-g)) = g \circ (-h) \circ (-g) \\ &\iff g \circ (-h) \circ (-g) \in L \otimes O(F). \end{aligned}$$

Therefore, conjugation by g defines an isomorphism between $\mathrm{Stab}_{(L,\circ)}(\gamma)$ and the subgroup of $L \otimes O(F)$ formed of elements h' such that $(-g) \circ (-h') \circ g \in L$. To conclude, it suffices to show that the latter set coincides with $\mathrm{Stab}_{(L \otimes O(F), \circ)}(m)$. Let $h' \in L \otimes O(F)$. We have:

$$\begin{aligned} h' \in \mathrm{Stab}_{(L \otimes O(F), \circ)}(m) &\iff \sigma(h) \circ m \circ (-h) = m \\ &\iff \sigma(h) \circ \sigma(g) \circ (-g) \circ (-h) = \sigma(g) \circ (-g) \\ &\iff (-g) \circ (-h) \circ g = \sigma((-g) \circ (-h) \circ g) \\ &\iff (-g) \circ (-h) \circ g \in L. \end{aligned}$$

☐

Example 3.5 (Artin–Schreier theory). The abelian Lie algebra $L = \mathbb{Z}/p\mathbb{Z}$ corresponds to the cyclic group $(L, \circ) = \mathbb{Z}/p\mathbb{Z}$. We have $L \otimes O(F) = F$. The action of $L \otimes O(F)$ on itself is given by $x.m = m + x^p - x$ for $m, x \in F$. We recover Artin–Schreier theory for $\mathbb{Z}/p\mathbb{Z}$-extensions.

Remark 3.6. Finite p-groups/Lie $\mathbb{Z}_p$-algebras can be replaced by pro-p-groups and profinite Lie $\mathbb{Z}_p$-algebras everywhere.

Remark 3.7. Let F be a local field of characteristic p. Using a "fundamental domain" to minimize the redundancy of the parametrizations above, Abrashkin describes an explicit isomorphism between:

- the quotient of the wild quotient of Γ_F by its p-th commutators;
- $(\mathcal{L}, \circ)$, where $\mathcal{L}$ is the quotient of the profinite free Lie $\mathbb{Z}_p$-algebra with countably many generators by its p-th commutators.

The main advantage of this description over the traditional description of Γ_F ([Koc67]) is that Abrashkin computes the ideals corresponding to the ramification filtration, giving us access to invariants like the discriminant.

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