RELATIVE DISCRIMINANT EXPONENTS OF HOMOMORPHISMS

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Let K be a local field with finite residue field, let Γ_K be its absolute Galois group equipped with the Krull topology and its ramification filtration Γ_K^v in the upper numbering. Consider an exact sequence of finite groups

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1.$$

Fix a continuous homomorphism $\rho \in \mathrm{Hom}(\Gamma_K, G)$. We define its *(relative) discriminant exponent* as

$$\delta(\rho) := |G| \int_{-1}^{+\infty} \left(1 - \frac{1}{|\rho(\Gamma_K^v)|}\right) \mathrm{d}v.$$

Consider the projected homomorphism $\overline{\rho} := \pi \circ \rho \in \mathrm{Hom}(\Gamma_K, H)$, let $\Gamma_{K'} := \ker \overline{\rho} = \rho^{-1}(N)$, and consider the restricted homomorphism $\widetilde{\rho} := \rho|_{\Gamma_{K'}} \in \mathrm{Hom}(\Gamma_{K'}, N)$. For any $v \geq -1$, we have $\pi(\rho(\Gamma_K^v)) = \overline{\rho}(\Gamma_K^v)$, and

$$\ker\left(\rho(\Gamma_K^v) \xrightarrow{\pi} \overline{\rho}(\Gamma_K^v)\right) = \rho\left(\Gamma_K^v \cap \ker \overline{\rho}\right) = \rho\left(\Gamma_K^v \cap \Gamma_{K'}\right) = \widetilde{\rho}\left(\Gamma_{K'}^{\psi_{K'|K}(v)}\right)$$

where the last equality comes from [Ser62, Chap. IV, §1, Prop. 2 and §3, Prop. 15] (it suffices to prove it at the level of finite extensions—we then convert to the lower numbering, intersect, and convert back). We sum this up by the short exact sequence

$$1 \longrightarrow \widetilde{\rho}\left(\Gamma_{K'}^{\psi_{K'|K}(v)}\right) \longrightarrow \rho(\Gamma_K^v) \longrightarrow \overline{\rho}(\Gamma_K^v) \longrightarrow 1.$$

We have

$$\frac{1}{|N|}\delta(\widetilde{\rho}) = \int_{-1}^{+\infty} \left(1 - \frac{1}{|\widetilde{\rho}(\Gamma_{K'}^v)|}\right) \mathrm{d}v$$

$$= \int_{-1}^{+\infty} \left(1 - \frac{1}{\left|\widetilde{\rho}(\Gamma_{K'}^{\psi_{K'|K}(v')})\right|}\right) \left[\overline{\rho}(\Gamma_K^0) : \overline{\rho}(\Gamma_K^{v'})\right] \mathrm{d}v'$$

$$\text{with } v' = \varphi_{K'|K}(v), \text{ cf. [Ser62, Chap. IV, §3]}$$

$$= \int_{-1}^{+\infty} \left(1 - \frac{\left|\overline{\rho}(\Gamma_K^{v'})\right|}{|\rho(\Gamma_K^{v'})|}\right) \frac{|\overline{\rho}(\Gamma_K^0)|}{|\overline{\rho}(\Gamma_K^{v'})|} \mathrm{d}v'$$

$$= |\overline{\rho}(\Gamma_K^0)| \int_{-1}^{+\infty} \left(\frac{1}{|\overline{\rho}(\Gamma_K^{v'})|} - \frac{1}{|\rho(\Gamma_K^{v'})|}\right) \mathrm{d}v'$$

$$= |\overline{\rho}(\Gamma_K^0)| \int_{-1}^{+\infty} \left[\left(1 - \frac{1}{|\rho(\Gamma_K^{v'})|}\right) - \left(1 - \frac{1}{|\overline{\rho}(\Gamma_K^{v'})|}\right)\right] \mathrm{d}v'$$

$$= |\overline{\rho}(\Gamma_K^0)| \cdot \left(\frac{1}{|G|}\delta(\rho) - \frac{1}{|H|}\delta(\overline{\rho})\right)$$

whence a "relative discriminant formula"

$$\frac{1}{|G|}\delta(\rho) = \frac{1}{|H|}\delta(\overline{\rho}) + \frac{1}{|N| \cdot |\overline{\rho}(\Gamma_K^0)|}\delta(\widetilde{\rho})$$

which can also be written as

$$\delta(\rho) = |N| \cdot \delta(\overline{\rho}) + \frac{|H|}{|\overline{\rho}(\Gamma_K^0)|} \cdot \delta(\widetilde{\rho}).$$

REFERENCES

[Ser62]   Jean-Pierre Serre. *Corps Locaux.* Hermann, Paris, 1962. ISBN: 978-2-7056-1296-2.