SURJECTING ONTO (PRO-)NILPOTENT GROUPS

In what follows, G is a group, and $\pi\colon G \twoheadrightarrow G^{\mathrm{ab}}$ is always the canonical surjection (of kernel $[G,G]$). Let us start with the following (obvious) observation:

Proposition 1. *Assume that G is solvable, and let H be a normal subgroup of G with $\pi(H) = G^{\mathrm{ab}}$. Then, $H = G$.*

Proof. The condition $\pi(H) = G^{\mathrm{ab}}$ means that $G = H[G,G]$. We have $(G/H)^{\mathrm{ab}} = G/(H[G,G]) = 0$, so $(G/H)' = [G/H, G/H] = \ker\big(G/H \twoheadrightarrow (G/H)^{\mathrm{ab}}\big) = G/H$, and then $(G/H)^{(n)} = G/H$ for all $n \geq 0$. But G/H is solvable, so $(G/H)^{(n)} = 0$ for large enough n, and therefore $G/H = 0$. □

We shall generalize it to non-normal subgroups for nilpotent G (cf. [MKS04, Lemma 5.9]).

Proposition 2. *Assume that G is nilpotent. Let $f\colon H \to G$ be a group homomorphism such that $\pi \circ f$ is surjective (onto G^{ab}). Then, f is surjective.*

Proof. We prove this by induction on the size of G. The case $G = 1$ is clear, so we assume that $G \neq 1$ and that the result holds for groups of size $< |G|$. Since G is non-trivial and nilpotent, its center is non-trivial, so $|G/Z(G)| < |G|$. The composite map $f'\colon H \xrightarrow{f} G \to (G/Z(G)) \twoheadrightarrow (G/Z(G))^{\mathrm{ab}}$ is surjective, as it factors as $H \overset{\pi\circ f}{\twoheadrightarrow} G^{\mathrm{ab}} \twoheadrightarrow (G/Z(G))^{\mathrm{ab}}$, so by the induction hypothesis f' is surjective (onto $G/Z(G)$), meaning that $G = f(H)Z(G)$. This implies that the map $[H,H] \to [G,G]$ induced by f is surjective: indeed, $[G,G] = [f(H)Z(G), f(H)Z(G)] = [f(H), f(H)] = f([H,H])$. Finally, the fact that $\pi \circ f$ is surjective means that $G = f(H)[G,G]$, so $G = f(H)f([H,H]) = f(H[H,H]) = f(H)$. □

Proposition 3. *Assume that G is pro-nilpotent. Let $f\colon H \to G$ be a continuous group homomorphism from a compact (e.g. profinite) topological group H, such that $\pi \circ f$ is surjective (onto G^{ab}). Then, f is surjective.*

Proof. Write $G = \varprojlim_n G_n$ where (G_n) is a projective system of nilpotent groups. Since f is continuous and H is compact, $f(H)$ is compact in the Hausdorff space G, so closed. Thus, it suffices to show that f has dense image, i.e., that the composite maps $H \xrightarrow{f} G \twoheadrightarrow G_n$ are all surjective. But this follows from Proposition 2. □

In particular (taking f to be the inclusion $H \hookrightarrow G$), any closed subgroup H of a pro-nilpotent group G that satisfies $\pi(H) = G^{\mathrm{ab}}$ equals G.

References

[MKS04] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory.* Dover Publications, Inc., Mineola, NY, 2nd edition, 2004.