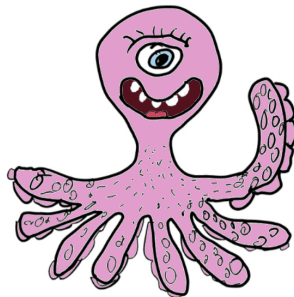


FROBENIUS & OCTOPUS



Béranger Seguin
(Universität Paderborn)

CARE, Bochum, Sep 2026

The problem

Let p be a prime, $q = p^d$

The absolute Frobenius σ acts on matrices **coefficientwise**

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

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Why do you even **CARE** ?

$\rightsquigarrow$ No time to explain today... (“number theory reasons”)

We focus on **semisimple matrices** (=diagonalizable over $\bar{\mathbb{F}}_p$).

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Define the following constructible subset of $\mathbb{A}^{n^2}$

$$X := \left\{ M \in \mathfrak{M}_n(\bar{\mathbb{F}}_p) \mid \begin{array}{l} M \text{ is diagonalizable} \\ M \text{ commutes with } \sigma(M) \end{array} \right\}$$

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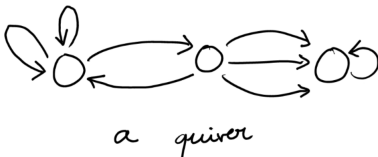
The question becomes...

Estimate $|X(\mathbb{F}_q)|$, i.e., compute $\dim X$.

(Lang–Weil says that $|X(\mathbb{F}_q)| \asymp q^{\dim X}$ for large q .)

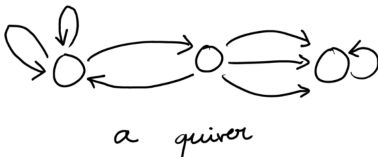
Quivers associated to matrices

A *quiver* is a directed graph where loops/parallel edges are allowed



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To each matrix $M \in \mathfrak{M}_n(\bar{\mathbb{F}}_p)$, with

$$\text{eigenvalues} = \lambda_1 \quad \dots \quad \lambda_r$$

$$\text{eigenspaces} = E_1 \quad \dots \quad E_r$$

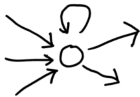
we associate a quiver $\mathcal{Q}(M)$ with

- **Vertices** = $\{1, \dots, r\}$
- **Number of edges** $i \rightarrow j = \dim(E_i \cap \sigma(E_j))$

Quiver criterion

Proposition

A matrix $M \in \mathfrak{M}_n(\bar{\mathbb{F}}_p)$ belongs to X (semi-simple, commutes with $\sigma(M)$) iff the quiver $\mathcal{Q}(M)$ has n edges and is *balanced*.



not balanced
(in-degree = 4)
(out-degree = 3)



balanced
(degree = 4)

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- If $M \in X$ then M and $\sigma(M)$ are co-diagonalizable so $E_i = \bigoplus_j E_i \cap \sigma(E_j)$, $\sigma(E_i) = \bigoplus_j E_j \cap \sigma(E_j)$, and $\bar{\mathbb{F}}_p^n = \bigoplus_i E_i$
- Conversely, using bases for $E_i \cap \sigma(E_j)$, we get a base diagonalizing both M and $\sigma(M)$

A stratification

$\text{Bal}_n =$ (finite) set of iso classes of balanced quivers with n edges

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We obtain a (locally closed) stratification:

$$X = \bigsqcup_{Q \in \text{Bal}_n} X_Q \quad \text{where } M \in X_Q \Leftrightarrow Q(M) \simeq Q$$

In particular $\dim X = \max_{Q \in \text{Bal}_n} \dim X_Q$.

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Question

What is the “generic” (dimension-maximizing) quiver?

The dimension of $X_{\mathcal{Q}}$

Let $\mathcal{Q}$ be a balanced quiver with

- n edges
- r vertices, say $\{1, \dots, r\}$
- $d_{i,j} :=$ number of edges $i \rightarrow j$
- $d_i := \sum_j d_{i,j}$ which is also $\sum_j d_{j,i}$ ($\mathcal{Q}$ is balanced)

We have $\sum d_i = n$ ($\mathcal{Q}$ has n edges).

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Proposition (Gundlach-S. 25)

$$\dim X_{\mathcal{Q}} = r + \sum_i d_i^2 - \sum_{i,j} d_{i,j}^2$$

$\rightsquigarrow$ a *purely combinatorial* formula in terms of the quiver $\mathcal{Q}$!

The dimension of X_Q — a lightning-fast explanation

- Choose eigenvalues (r distinct elements of $\bar{\mathbb{F}}_p$) $\rightsquigarrow$ the term r
- It remains to pick linearly independent spaces $F_{i,j} = E_i \cap \sigma(E_j)$ of dimension $d_{i,j}$ with $\bigoplus_j F_{j,i} = \sigma\left(\bigoplus_j F_{i,j}\right)$
- Instead, we pick arbitrary bases for each $F_{i,j}$
 - $\rightsquigarrow$ artificially increases dimension by $\dim\left(\prod_{i,j} \mathrm{GL}_{d_{i,j}}(\bar{\mathbb{F}}_p)\right)$
 - $\rightsquigarrow$ explains the negative term $-\sum_{i,j} d_{i,j}^2$
- $F_{i,j}$ will generically be in direct sum ($\det \neq 0$) so assume this
- Choice of basis $\rightsquigarrow$ given by a matrix $F \in \mathrm{GL}_n(\bar{\mathbb{F}}_p)$

In terms of the matrix $\wp(F) := F^{-1}\sigma(F)$, the condition “ $\bigoplus_j F_{j,i} = \sigma\left(\bigoplus_j F_{i,j}\right)$ ” becomes “ $\wp(F)$ preserves $\bigoplus_j F_{j,i}$ ”, so we are counting block-diagonal matrices

 - $\rightsquigarrow$ dimension $\sum_i d_i^2$ for the choice of $\wp(F)$
 - $\rightsquigarrow$ same dim for the choice of F because $\wp$ is an étale map

$$(d\wp = M^{-1} dM M^{-1} \sigma(M) \text{ is an linear iso in the variable } dM)$$

Equivalent problem

Funny (nontrivial) exercise in combinatorics:

Find $Q \in \text{Bal}_n$ maximizing

$$r + \sum_i d_i^2 - \sum_{i,j} d_{i,j}^2$$

where

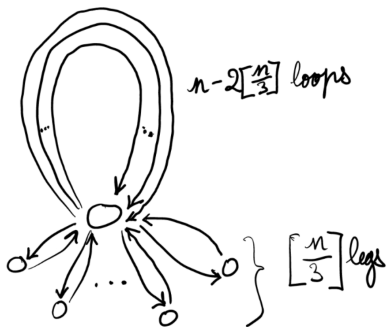
r = nb. of vertices

$d_{i,j}$ = nb. of edges $i \rightarrow j$

$$d_i = \sum_j d_{i,j} = \sum_j d_{j,i} = \text{degree of } i$$

The octopus quiver

$\lceil n/3 \rceil = \text{integer closest to } n/3$



Total : n edges, $\lceil \frac{n}{3} \rceil + 1$ vertices

the octopus quiver

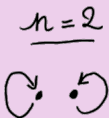
From Greek

πούς "foot"
οκτώ = $\lceil \frac{n}{3} \rceil$

Maximizing quivers

Theorem (Gundlach-S. 25)

The maximal value of $\dim X_{\mathcal{Q}}$ is $\lfloor n^2/3 \rfloor + 1$. It is obtained if and only if $\mathcal{Q}$ is the octopus quiver, except in these two cases:



my coauthor refused to name it the *Yedynauka* quiver! ☹

Corollary (Gundlach-S. 25)

We have $|X(\mathbb{F}_q)| \asymp q^{\lfloor n^2/3 \rfloor + 1}$.

We also compute the leading coefficient, which is 1 if $n \geq 5$.

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Thanks for your attention!