Asymptotics of extensions of function fields

Béranger Seguin

Institut für Mathematik
University of Paderborn

Conference *Arithmetic Statistics*
University of Paderborn
August 24th, 2026

Outline

I. Extensions, and how to measure them

II. Let's start counting: tame extensions

III. Counting (potentially wild) abelian extensions

IV. Non-abelian wild extensions

V. Things to think about...

Extensions

K a field, Γ_K its absolute Galois group, G a finite group.

Definition

G-extension of K = étale K-algebra with action of G, that becomes isomorphic over $\bar{K}$ to the algebra of maps $G \to \bar{K}$

Extensions

K a field, Γ_K its absolute Galois group, G a finite group.

Definition

G-extension of K = étale K-algebra with action of G, that becomes isomorphic over $\bar{K}$ to the algebra of maps $G \to \bar{K}$

Proposition

$$\{G\text{-extensions of } K\}/\text{isom.} \overset{\text{bij}}{\longleftrightarrow} \mathrm{Hom}(\Gamma_K, G)/\text{conj.}$$

where $\mathrm{Hom}(\Gamma_K, G)$ = continuous group homomorphisms $\Gamma_K \to G$

Note that $\mathrm{Hom}(\Gamma_K, G)/\text{conj.} = H^1(K, G)$.

Extensions

K a field, Γ_K its absolute Galois group, G a finite group.

Definition

G-extension of K = étale K-algebra with action of G, that becomes isomorphic over $\bar{K}$ to the algebra of maps $G \to \bar{K}$

Proposition

$$\{G\text{-extensions of } K\}/\text{isom.} \xleftrightarrow{\text{bij}} \mathrm{Hom}(\Gamma_K, G)/\text{conj.}$$

where $\mathrm{Hom}(\Gamma_K, G)$ = continuous group homomorphisms $\Gamma_K \to G$

Note that $\mathrm{Hom}(\Gamma_K, G)/\text{conj.} = H^1(K, G)$.

Example

Any Galois extension $L|K$ with an isomorphism $\mathrm{Gal}(L|K) \cong G$ is a G-extension ($\leftrightsquigarrow$ surjective homomorphisms)

Ramification filtration 1/2

K a local field of residue characteristic p

Γ_K admits a decreasing ramification filtration $(\Gamma_K^v)_{v \in \mathbb{R}_{\geq -1}}$

$$\Gamma_K^{-1} \trianglerighteq \Gamma_K^0 \trianglerighteq \ldots \trianglerighteq \Gamma_K^1 \trianglerighteq \ldots \trianglerighteq \Gamma_K^2 \trianglerighteq \ldots$$

Ramification filtration 1/2

K a local field of residue characteristic p

Γ_K admits a decreasing ramification filtration $(\Gamma_K^v)_{v \in \mathbb{R}_{\geq -1}}$

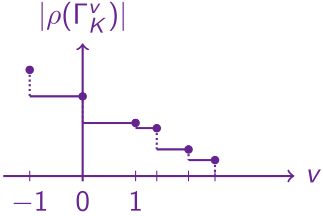
$$\Gamma_K^{-1} \trianglerighteq \Gamma_K^0 \trianglerighteq \ldots \trianglerighteq \Gamma_K^1 \trianglerighteq \ldots \trianglerighteq \Gamma_K^2 \trianglerighteq \ldots$$

A few facts:

- All the Γ_K^v are normal subgroups of $\Gamma_K = \Gamma_K^{-1}$
- $\Gamma_K^0 =$ inertia subgroup (so $\Gamma_K^{-1}/\Gamma_K^0 = \hat{\mathbb{Z}}$)
- $\bigcup_{v>0} \Gamma_K^v =$ wild inertia subgroup (unique pro-p-Sylow of Γ_K^0)
- Γ_K^v for $v > 0 \rightsquigarrow$ higher inertia groups (pro-p-groups)
- $\bigcap_{v \geq -1} \Gamma_K^v = 1$

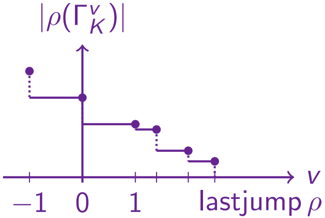
K a local field of residue characteristic p

For any $\rho\colon \Gamma_K \to G$, we have the filtration $G^v = \rho(\Gamma_K^v)$



K a local field of residue characteristic p

For any $\rho\colon \Gamma_K \to G$, we have the filtration $G^v = \rho(\Gamma_K^v)$



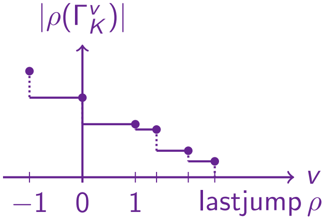
An invariant/height:

$$\operatorname{lastjump}\rho := \inf \left\{ \begin{matrix} v \geq 0 : \\ \rho(\Gamma_K^v) = 1 \end{matrix} \right\}$$

Vanishes iff ρ tamely ramified

K a local field of residue characteristic p

For any $\rho\colon \Gamma_K \to G$, we have the filtration $G^v = \rho(\Gamma_K^v)$



An invariant/height:

$$\mathrm{lastjump}\,\rho := \inf \left\{ \begin{array}{c} v \geq 0 : \\ \rho(\Gamma_K^v) = 1 \end{array} \right\}$$

Vanishes iff ρ tamely ramified

Discriminant exponent

$$|G| \int_{-1}^{+\infty} \left(1 - \frac{1}{|\rho(\Gamma_K^v)|}\right) \mathrm{d}v$$

Artin conductor of a character χ

$$\int_{-1}^{+\infty} \left(\chi(1) - \chi(G^v)\right) \mathrm{d}v$$

Last jump in lower numbering

$$\mathrm{lastjump}_{\mathrm{low}}\,\rho$$

Turning local invariants into global invariants

K a global function field, $\mathcal{P}$ the set of its places, and inv some local invariant. For any $\rho\colon \Gamma_K \to G$, let

$$\operatorname{inv} \rho := \sum_{\mathfrak{p} \in \mathcal{P}} \deg \mathfrak{p} \cdot \operatorname{inv}(\rho|_{\Gamma_{K_\mathfrak{p}}})$$

Turning local invariants into global invariants

K a global function field, $\mathcal{P}$ the set of its places, and inv some local invariant. For any $\rho\colon \Gamma_K \to G$, let

$$\operatorname{inv}\rho := \sum_{\mathfrak{p}\in\mathcal{P}} \deg\mathfrak{p}\cdot\operatorname{inv}(\rho|_{\Gamma_{K_\mathfrak{p}}})$$

(Over number fields, use a "multiplicative invariant" like $\prod_\mathfrak{p} \mathrm{Nm}(\mathfrak{p})^{\operatorname{inv}_\mathfrak{p}\rho}$.)

Turning local invariants into global invariants

K a global function field, $\mathcal{P}$ the set of its places, and inv some local invariant. For any $\rho\colon \Gamma_K \to G$, let

$$\operatorname{inv} \rho := \sum_{\mathfrak{p} \in \mathcal{P}} \deg \mathfrak{p} \cdot \operatorname{inv}(\rho|_{\Gamma_{K_{\mathfrak{p}}}})$$

(Over number fields, use a "multiplicative invariant" like $\prod_{\mathfrak{p}} \mathrm{Nm}(\mathfrak{p})^{\operatorname{inv}_{\mathfrak{p}} \rho}$.)

Hermite–Minkowski-type theorems (Northcott property)

$$\{\rho\colon \Gamma_K \to G \mid \operatorname{inv} \rho \leq X\} \text{ is finite for all } X$$

Turning local invariants into global invariants

K a global function field, $\mathcal{P}$ the set of its places, and inv some local invariant. For any $\rho\colon \Gamma_K \to G$, let

$$\operatorname{inv} \rho := \sum_{\mathfrak{p} \in \mathcal{P}} \deg \mathfrak{p} \cdot \operatorname{inv}(\rho|_{\Gamma_{K_\mathfrak{p}}})$$

(Over number fields, use a "multiplicative invariant" like $\prod_\mathfrak{p} \mathrm{Nm}(\mathfrak{p})^{\operatorname{inv}_\mathfrak{p} \rho}$.)

Hermite–Minkowski-type theorems (Northcott property)

$$\{\rho\colon \Gamma_K \to G \mid \operatorname{inv} \rho \leq X\} \text{ is finite for all } X$$

Malle-type questions (effective Hermite–Minkowski)

$$\text{Asymptotics of } |\{\rho\colon \Gamma_K \to G \mid \operatorname{inv} \rho \leq X\}| \text{ as } X \to \infty?$$

Geometric point of view

- A global function field K with field of constants $\mathbb{F}_q$
  $\leftrightsquigarrow$ a curve X over $\mathbb{F}_q$
- A G-extension $L|K$ $\leftrightsquigarrow$ a Galois G-cover $Y \to X$
  (étale extensions that are not fields $\leftrightsquigarrow$ non-connected Y)
- Discriminant of $L|K \approx$ Genus of Y (by Riemann-Hurwitz)

Geometric point of view

- A global function field K with field of constants $\mathbb{F}_q$
  $\leftrightsquigarrow$ a curve X over $\mathbb{F}_q$
- A G-extension $L|K \leftrightsquigarrow$ a Galois G-cover $Y \to X$
  (étale extensions that are not fields $\leftrightsquigarrow$ non-connected Y)
- Discriminant of $L|K \approx$ Genus of Y (by Riemann-Hurwitz)

Hence:

Statistics of G-extensions of K ordered by discriminant
$\leftrightsquigarrow$ Statistics of G-covers of X ordered by genus

Tame extensions of local function fields

G a finite group, $K = \mathbb{F}_q(\!(T)\!)$ a local function field

Tame extensions of local function fields

G a finite group, $K = \mathbb{F}_q(\!(T)\!)$ a local function field

- There are finitely many tame G-extensions of K
  (pick a "generator of inertia" of order coprime-to-p, and a "Frobenius element" + mild conditions)

Tame extensions of local function fields

G a finite group, $K = \mathbb{F}_q(\!(T)\!)$ a local function field

- There are finitely many tame G-extensions of K (pick a "generator of inertia" of order coprime-to-p, and a "Frobenius element" $+$ mild conditions)
- **Asymptotics:** some constant...

$\rightsquigarrow$ Over local function fields, nothing interesting happens when counting tame extensions!

Tame extensions of local function fields

G a finite group, $K = \mathbb{F}_q(\!(T)\!)$ a local function field

- There are finitely many tame G-extensions of K (pick a "generator of inertia" of order coprime-to-p, and a "Frobenius element" $+$ mild conditions)
- **Asymptotics:** some constant...

$\rightsquigarrow$ Over local function fields, nothing interesting happens when counting tame extensions!

(Over p-adic fields, the same is true even for wild extensions, but you have to count "Demuškin tuples" within your group)

Tame extensions of global function fields 1/2

The case $K = \mathbb{F}_q(T)$: close to number fields/Malle's conjecture.

Hurwitz spaces $=$ moduli spaces of tame G-covers of $\mathbb{P}^1$

- Their $\mathbb{F}_q$-points $\approx$ tame G-extensions of $\mathbb{F}_q(T)$

Tame extensions of global function fields 1/2

The case $K = \mathbb{F}_q(T)$: close to number fields/Malle's conjecture.

Hurwitz spaces = moduli spaces of tame G-covers of $\mathbb{P}^1$

- Their $\mathbb{F}_q$-points $\approx$ tame G-extensions of $\mathbb{F}_q(T)$
- Use Grothendieck–Lefschetz–Behrend trace formulas + cohomological methods to count $\mathbb{F}_q$-points for large q

Tame extensions of global function fields 1/2

The case $K = \mathbb{F}_q(T)$: close to number fields/Malle's conjecture.

Hurwitz spaces = moduli spaces of tame G-covers of $\mathbb{P}^1$

- Their $\mathbb{F}_q$-points $\approx$ tame G-extensions of $\mathbb{F}_q(T)$
- Use Grothendieck–Lefschetz–Behrend trace formulas + cohomological methods to count $\mathbb{F}_q$-points for large q
- Main problem: prove homological stability.
  An algebraic topology question!

Tame extensions of global function fields 2/2

Thanks to the Hurwitz approach, the case $K = \mathbb{F}_q(T)$ is now **mostly solved** for G coprime to p and large q!

- Ellenberg–Venkatesh '05, Türkelli '15, E–V–Westerland '16, E–Tran–W '17, **Landesman–Levy '25**, Santens '25
- $+$ results on coprime-to-p part of class groups (Sawin-Wood).

Tame extensions of global function fields 2/2

Thanks to the Hurwitz approach, the case $K = \mathbb{F}_q(T)$ is now **mostly solved** for G coprime to p and large q!

- Ellenberg–Venkatesh '05, Türkelli '15, E–V–Westerland '16, E–Tran–W '17, **Landesman–Levy '25**, Santens '25
- $+$ results on coprime-to-p part of class groups (Sawin-Wood).

Theorem

Count of G-extensions of $\mathbb{F}_q(T)$ with discriminant exponent $\leq X$:

$$\rightsquigarrow C(X) \cdot q^{X/a} \cdot X^{b-1}$$

where a is as predicted by Malle, b is as predicted by Türkelli, and the "constant" C is described by Santens.

($\rightarrow$ Tim's talk!)

Remark

For tame $L|K$, any unramified p-extension of L that is Galois over K is also a tame extension of K.

Hurwitz spaces do not just parametrize coprime-to-p covers, but all tame G-covers.

$\rightsquigarrow$ What can Hurwitz methods say about p-parts of class groups?

The general abelian case

- **Tame abelian extensions of global function fields** $\rightsquigarrow$ asymptotics given by Wright '89 via class field theory (CFT)

The general abelian case

- **Tame abelian extensions of global function fields** $\rightsquigarrow$ asymptotics given by Wright '89 via class field theory (CFT)
- **General abelian extensions** $\rightsquigarrow$ also theoretically parametrized by CFT, but combinatorial/analytic difficulties. (Additional tool: **Artin-Schreier-Witt theory** gives an explicit parametrization of cyclic p-extensions.)

Remark

For wild extensions, asymptotics are interesting even over local function fields!

The general abelian case

- **Tame abelian extensions of global function fields** $\rightsquigarrow$ asymptotics given by Wright '89 via class field theory (CFT)
- **General abelian extensions** $\rightsquigarrow$ also theoretically parametrized by CFT, but combinatorial/analytic difficulties. (Additional tool: **Artin-Schreier-Witt theory** gives an explicit parametrization of cyclic p-extensions.)

Remark

For wild extensions, asymptotics are interesting even over local function fields!

- Although moduli spaces for wild covers are constructed in special cases (cf. Rachel Pries' work), they are not used as in the tame case at the moment.

Recent work on the wild abelian case

The wild abelian case is now essentially complete:

- We can count by [conductor/discriminant/last jump] over [local/global] function fields
- Lagemann '10-'12'-'15, Klüners-Müller '20, Potthast '24, Gundlach '25-'26, Jiazhi He '26, Bhatnagar-Potthast-Yin ($\to$ Nicolas' talk!)

Recent work on the wild abelian case

The wild abelian case is now essentially complete:

- We can count by [conductor/discriminant/last jump] over [local/global] function fields
- Lagemann '10-'12'-'15, Klüners-Müller '20, Potthast '24, Gundlach '25-'26, Jiazhi He '26, Bhatnagar-Potthast-Yin ($\rightarrow$ Nicolas' talk!)
- Asymptotics are generally **not consistent** with Malle's conjecture (even the a constant)

Recent work on the wild abelian case

The wild abelian case is now essentially complete:

- We can count by [conductor/discriminant/last jump] over [local/global] function fields
- Lagemann '10-'12'-'15, Klüners-Müller '20, Potthast '24, Gundlach '25-'26, Jiazhi He '26, Bhatnagar-Potthast-Yin ($\rightarrow$ Nicolas' talk!)
- Asymptotics are generally **not consistent** with Malle's conjecture (even the a constant)
- Unexpected: Gundlach shows that the generating function

$$\sum_{\rho\colon \Gamma_K \to G} X^{\mathrm{lastjump}\,\rho}$$

is a rational function of X (for $K|\mathbb{F}_p(T)$ finite, G abelian p-group)

Non-abelian wild extensions

We focus on p-groups $\rightsquigarrow$ no tame ramification.

A *terra incognita*: no real conjecture at the moment.

Non-abelian wild extensions

We focus on p-groups $\rightsquigarrow$ no tame ramification.

A *terra incognita*: no real conjecture at the moment.

Remark

In the local case $K = \mathbb{F}_q(\!(T)\!)$, even though Γ_K is known as a topological group (Koch: wild inertia is free pro-p with ω generators), this says nothing of the ramification filtration (thus nothing on discriminants, last jumps, etc.).
Therefore, the resulting "parametrization" is useless for counting.

$\rightsquigarrow$ even the local case is hard

A useful generalization of Artin-Schreier theory

K a field of characteristic p.

$G = \mathcal{G}(\mathbb{F}_p)$ a finite group, for an $\mathbb{F}_p$-group scheme $\mathcal{G}$ satisfying

- $H^1(K, \mathcal{G}(\bar{K}))$ is trivial ("Hilbert 90")
- the map $\wp\colon \mathcal{G}(\bar{K}) \to \mathcal{G}(\bar{K})$, $g \mapsto \sigma(g)g^{-1}$ is surjective (σ is the absolute Frobenius induced by $x \mapsto x^p$)

(e.g. if $\mathcal{G}$ is unipotent, or decomposes into factors that are all $\mathbb{G}_a$, $\mathbb{G}_m$ or GL_n via short exact sequences)

A useful generalization of Artin-Schreier theory

K a field of characteristic p.

$G = \mathcal{G}(\mathbb{F}_p)$ a finite group, for an $\mathbb{F}_p$-group scheme $\mathcal{G}$ satisfying

- $H^1(K, \mathcal{G}(\bar{K}))$ is trivial ("Hilbert 90")
- the map $\wp\colon \mathcal{G}(\bar{K}) \to \mathcal{G}(\bar{K})$, $g \mapsto \sigma(g)g^{-1}$ is surjective
  (σ is the absolute Frobenius induced by $x \mapsto x^p$)

(e.g. if $\mathcal{G}$ is unipotent, or decomposes into factors that are all $\mathbb{G}_a$, $\mathbb{G}_m$ or GL_n via short exact sequences)

Theorem (parametrization of G-extensions)

$$H^1(K, G) \overset{\text{bij}}{\longleftrightarrow} \mathcal{G}(K)//\mathcal{G}(K)$$

where $H^1(K, G) =$ G-extensions of K

$\mathcal{G}(K)//\mathcal{G}(K) =$ orbits for σ-conjugation $g.m = \sigma(g)mg^{-1}$

Nilpotent Artin-Schreier theory

The previous theorem is not a miracle tool: apparently, not every p-group is $\mathcal{G}(\mathbb{F}_p)$ for some unipotent $\mathcal{G}$

(someone on `mathoverflow` claims that D_{16} is a counterexample for $p = 2$)

The previous theorem is not a miracle tool: apparently, not every p-group is $\mathcal{G}(\mathbb{F}_p)$ for some unipotent $\mathcal{G}$

(someone on `mathoverflow` claims that D_{16} is a counterexample for $p = 2$)

HOWEVER! When G has nilpotency class $< p$, we have $G = \mathcal{G}(\mathbb{F}_p)$ for a unipotent $\mathcal{G}$, with $G \mapsto \mathcal{G}$ an exact functor.

The previous theorem is not a miracle tool: apparently, not every p-group is $\mathcal{G}(\mathbb{F}_p)$ for some unipotent $\mathcal{G}$

(someone on `mathoverflow` claims that D_{16} is a counterexample for $p = 2$)

HOWEVER! When G has nilpotency class $< p$, we have $G = \mathcal{G}(\mathbb{F}_p)$ for a unipotent $\mathcal{G}$, with $G \mapsto \mathcal{G}$ an exact functor.

($\mathcal{G}$ is constructed using the Lazard correspondence: to define $\mathcal{G}(K)$, turn G into a Lie algebra, base change it to K, and turn the Lie algebra back into a group)

The previous theorem is not a miracle tool: apparently, not every p-group is $\mathcal{G}(\mathbb{F}_p)$ for some unipotent $\mathcal{G}$

(someone on `mathoverflow` claims that D_{16} is a counterexample for $p = 2$)

HOWEVER! When G has nilpotency class $< p$, we have $G = \mathcal{G}(\mathbb{F}_p)$ for a unipotent $\mathcal{G}$, with $G \mapsto \mathcal{G}$ an exact functor.

($\mathcal{G}$ is constructed using the Lazard correspondence: to define $\mathcal{G}(K)$, turn G into a Lie algebra, base change it to K, and turn the Lie algebra back into a group)

This is Abrashkin's **nilpotent Artin-Schreier theory.**

- $\rightsquigarrow$ yields a description of $\Gamma_K^{\text{pro-}p}$ modulo p-th commutators
- when $K = \mathbb{F}_q(\!(T)\!)$, Abrashkin also describes the ramification filtration in terms of this description!

Local estimates

$K = \mathbb{F}_q(\!(T)\!)$ of characteristic $p \neq 2$
G a p-group of nilpotency class ≤ 2, i.e., $[G, [G, G]] = 1$

Using the parametrization of G-extensions + Abrashkin's description of the ramification filtration, Gundlach and I estimate

$$|\{\rho \colon \Gamma_{\mathbb{F}_q(\!(T)\!)} \to G \mid \operatorname{lastjump} \rho = X\}|$$

Local estimates

$K = \mathbb{F}_q(\!(T)\!)$ of characteristic $p \neq 2$
G a p-group of nilpotency class ≤ 2, i.e., $[G,[G,G]] = 1$

Using the parametrization of G-extensions + Abrashkin's description of the ramification filtration, Gundlach and I estimate

$$|\{\rho\colon \Gamma_{\mathbb{F}_q(\!(T)\!)} \to G \mid \operatorname{lastjump} \rho = X\}|$$

We do not actually obtain asymptotics, but

- precise estimates for small X
- upper bounds for large X

Local estimates

$K = \mathbb{F}_q(\!(T)\!)$ of characteristic $p \neq 2$
G a p-group of nilpotency class ≤ 2, i.e., $[G, [G, G]] = 1$

Using the parametrization of G-extensions + Abrashkin's description of the ramification filtration, Gundlach and I estimate

$$|\{\rho\colon \Gamma_{\mathbb{F}_q(\!(T)\!)} \to G \mid \operatorname{lastjump} \rho = X\}|$$

We do not actually obtain asymptotics, but

- precise estimates for small X
- upper bounds for large X

Turns out to be enough for global asymptotics...

Question

Could the local counting problem be harder than the global one?

Local-global principle

Let $K = \mathbb{F}_q(T)$ of characteristic p
G a p-group of nilpotency class ≤ 2.

Gundlach and I prove an **exact quantitative local-global principle** for G-extensions counted by last jump:

Theorem (Gundlach-S. '25-'26)

$$\frac{1}{|G|}\left|\left\{\begin{matrix}\rho\colon \Gamma_K \to G \\ \forall \mathfrak{p} \in \mathcal{P},\ \mathrm{lastjump}(\rho|_{\Gamma_{K_\mathfrak{p}}}) = X_\mathfrak{p}\end{matrix}\right\}\right|$$
$$= \prod_{\mathfrak{p}\in\mathcal{P}} \frac{1}{|G|}\left|\left\{\begin{matrix}\rho_\mathfrak{p}\colon \Gamma_{K_\mathfrak{p}} \to G \\ \mathrm{lastjump}(\rho_\mathfrak{p}) = X_\mathfrak{p}\end{matrix}\right\}\right|$$

(simplified and extended to $p = 2$ in a later paper)

Local estimates + local–global principle ⇒ **global asymptotics**

Theorem (Gundlach-S. '25)

Let $K = \mathbb{F}_q(T)$ of characteristic $p \neq 2$. Let G be a p-group of nilpotency class ≤ 2. Assume that the p-torsion subgroup $G[p]$ is either abelian or of size $\leq p^{p-1}$, and let

$$r := \log_p |G[p]| \qquad\qquad M := \begin{cases} 1 & \text{if } G[p] \text{ is abelian} \\ 1 + \frac{1}{p} & \text{otherwise.} \end{cases}$$

Assume that $G[p] = G$ or that q is large enough. Then, for some M-periodic function $C \colon \mathbb{Q}/M\mathbb{Z} \to \mathbb{R}_{\geq 0}$ with $C(0) \neq 0$, we have

$$\frac{1}{|G|} \left| \left\{ \begin{array}{c} \rho \colon \Gamma_K \to G \\ \operatorname{lastjump} \rho = X \end{array} \right\} \right| = C(X) \cdot q^{\frac{r+1}{M} \cdot X} + o\left(q^{\frac{r+1}{M} \cdot X} \right)$$

Mixing wild and tame

- In ongoing work, I study $G = C_p \rtimes C_{p-1}$ in characteristic p
- $\mathcal{G} = \mathbb{G}_a \rtimes \mathbb{G}_m \rightsquigarrow$ a parametrization of G-extensions
- I give exact local counts when counting by discriminant
- No exact local-global principle here
  $\rightsquigarrow$ we need some control of class groups for global counts...

Mixing wild and tame

- In ongoing work, I study $G = C_p \rtimes C_{p-1}$ in characteristic p
- $\mathcal{G} = \mathbb{G}_a \rtimes \mathbb{G}_m \rightsquigarrow$ a parametrization of G-extensions
- I give exact local counts when counting by discriminant
- No exact local-global principle here
  $\rightsquigarrow$ we need some control of class groups for global counts...

To be continued!

I. Extensions, and how to measure them

II. Let's start counting: tame extensions

III. Counting (potentially wild) abelian extensions

IV. Non-abelian wild extensions

V. Things to think about...

Things to think about...

- Do we start noticing patterns? Could we make a conjecture?

  (Darda–Yasuda have a conjecture which would allow to deduce the global asymptotics from local dimensions/counts. Compatible with an "approximate local-global principle".)

- Could moduli (or stacky) methods work here?
  How to reconcile the wild methods with the tame methods to tackle hybrid cases?

Thanks for your attention!