Remarks on Fadelian Rings

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1 The Weak Ore Property and Fadelianity (2022-05-12)

Let R be an integral ring.

Definition 1.1. *We define the following subsets of R:*

- $\mathcal{F}$ *is the set of* fadelian elements*:*

$$\mathcal{F} = \{x \in R \,|\, \forall a \in R \setminus \{0\}, \exists b, c \in R,\, x = ab + ca\}\,;$$

- $\mathcal{L}$ *is the set of* left Ore elements*:*

$$\mathcal{L} = \{x \in R \,|\, \forall a \in R \setminus \{0\}, \exists b, c \in R \setminus \{0\},\, ba = cx\}\,;$$

- $\mathcal{R}$ *is the set of* right Ore elements*:*

$$\mathcal{R} = \{x \in R \,|\, \forall a \in R \setminus \{0\}, \exists b, c \in R \setminus \{0\},\, ab = xc\}\,;$$

Proposition 1.2. *We claim the following:*

- *$\mathcal{L}$, $\mathcal{R}$ and $\mathcal{F}$ are stable by multiplication by an element of $Z(R)$.*
- *$\mathcal{L}$ and $\mathcal{R}$ contain units of R;*
- *$\mathcal{L}$ and $\mathcal{R}$ are multiplicatively closed;*
- *$\mathcal{F}$ is additively closed.*

Proof. We prove the statements in order:

- This is clear;
- If $x \in R^\times$ and $a \neq 0$, then $a \cdot 1 = x \cdot (x^{-1}a)$ and $1 \cdot a = (ax^{-1}) \cdot x$.
- Let's prove the claim for $\mathcal{L}$. Let $x, y \in \mathcal{L}$ and $a \neq 0$. Choose nonzero b, c such that $ba = cy$. Now choose nonzero b', c' such that $b'c = c'x$. Then:

$$c'xy = b'cy = b'ba.$$

- If $x = ab + ca$ and $x' = ab' + c'a$ then $x + x' = a(b + b') + (c + c')a$.

□

Proposition 1.3. *$\mathcal{F}$ is stable by left-mutliplication with an element of $\mathcal{R}$, and by right-multiplication with an element of $\mathcal{L}$.*

Proof. Let $w = xyz$ with $x \in \mathcal{R}, y \in \mathcal{F}, z \in \mathcal{L}$. Consider $a \in R \setminus \{0\}$. Fix $b, c \neq 0$ such that $ab = xc$ and $b', c' \neq 0$ such that $b'a = c'z$. Now choose u, v such that:

$$y = ucc' + cc'v.$$

We have:

$$w = xyz = xucc'z + xcc'vz = xucb'a + abc'vz \in Ra + aR.$$

This proves the result. □

Definition 1.4. *We say that R satisfies the weak Ore property if every element x may be written as a sum:*

$$x = r_1 l_1 + \ldots + r_n l_n$$

where $r_i \in \mathcal{R}$ and $l_i \in \mathcal{L}$.

For example, if R is Ore or additively generated by its units, it satisfies the weak Ore property.

Theorem 1.5. *If R is weakly fadelian and satisfies the weak Ore property, then R is fadelian.*

Proof. Since R is weakly fadelian, we have $1 \in \mathcal{F}$, and thus $\mathcal{R}\mathcal{L} \subseteq \mathcal{F}$ by proposition 1.3. Since $\mathcal{F}$ is additively closed, all elements of the form $\sum r_i l_i$ with $r_i \in \mathcal{R}, l_i \in \mathcal{L}$ are in $\mathcal{F}$. By the weak Ore property, we have:

$$\mathcal{F} = R.$$

This means that R is fadelian. □