Flimsy Spaces

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Definition. Let $n \geq 1$. A connected topological space X is said to be n-flimsy if removing fewer then n arbitrary points leaves the space connected and removing any n arbitrary (distinct) points disconnects the space. X has to have more than n points.

For example, $\mathbb{R}$ is 1-flimsy and S^1 is 2-flimsy. In the following, we prove 3-flimsy spaces do not exist.

Theorem 1. *Let X a 2-flimsy space and $x, y \in X$, with $x \neq y$. $X \backslash \{x, y\}$ has exactly two connected components.*

Proof. It is equivalent to show that $X \backslash \{x, y\}$ can not be covered by three disjoint non-empty open sets. Let be three open sets of X, U_1, U_2, and U_3, such that $(U_1 \cup U_2 \cup U_3) \cap \{x, y\}^c = X \backslash \{x, y\}$ and $U_1 \cap U_2 \cap \{x, y\}^c = U_1 \cap U_3 \cap \{x, y\}^c = U_2 \cap U_3 \cap \{x, y\}^c = \emptyset$. We are going to show by contradiction that there is $i \in \{1, 2, 3\}$ such that $U_i \cap \{x, y\}^c = \emptyset$: let us suppose $\forall i \in \{1, 2, 3\},\ U_i \cap \{x, y\}^c \neq \emptyset$.

We choose $u_1 \in U_1 \cap \{x, y\}^c$ and $u_2 \in U_2 \cap \{x, y\}^c$, so $u_1 \notin U_2 \cup U_3$ and $u_2 \notin U_1 \cup U_3$. We are going to prove $X \backslash \{u_1, u_2\}$ is connected, which contradicts that X is 2-flimsy.

Let U, V two open sets of X such that $(U \cup V) \cap \{u_1, u_2\}^c = X \backslash \{u_1, u_2\}$ and $U \cap V \cap \{u_1, u_2\}^c = \emptyset$. We can suppose $x \in U$ without loss of generality, and so $x \notin V$.

1. $U \cup U_1 \cup U_2$ and $V \cap U_3$ are open.

   $(U \cup U_1 \cup U_2) \cap (V \cap U_3) \subset (U \cap V) \cup (U_1 \cap U_3) \cup (U_2 \cap U_3) \subset \{u_1, u_2, x, y\}$ but $x \notin V$, and $u_1, u_2 \notin U_3$ so $(U \cup U_1 \cup U_2) \cap (V \cap U_3) \cap \{y\}^c = \emptyset$

   $(U \cup U_1 \cup U_2) \cup (V \cap U_3) \supset U_1 \cup U_2 \cup (U_3 \cap (U \cup V)) \supset (U_1 \cup U_2 \cup U_3) \cap \{u_1, u_2\}^c \supset X \backslash \{u_1, u_2, x, y\}$ but $x \in U$, $u_1 \in U_1$, and $u_2 \in U_2$ so $((U \cup U_1 \cup U_2) \cup (V \cap U_3)) \cap \{y\}^c = X \backslash \{y\}$

   X is 2-flimsy so $X \backslash \{y\}$ is connected. Moreover $x \in (U \cup U_1 \cup U_2) \cap \{y\}^c \neq \emptyset$.

   So $(V \cap U_3) \cap \{y\}^c = \emptyset$

2. If $y \in V$, then $y \notin U$ and the previous step implies $(U \cap U_3) \cap \{x\}^c = \emptyset$. Then $U_3 \cap \{x, y\}^c \subset (U_3 \cap U \cap \{x\}^c) \cup (U_3 \cap V \cap \{y\}^c) \cup (U_3 \cap \{u_1, u_2\}) = \emptyset$ which is false.

   So $y \in U$, $y \notin V$, $V \cap U_3 = \emptyset$, and $U_3 \subset U$

3. $U \cup U_1$ and $V \cap U_2$ are open.

   $(U \cup U_1) \cap (V \cap U_2) \subset (U \cap V) \cup (U_1 \cap U_2) \subset \{x, y, u_1, u_2\}$ but $u_1 \notin U_2$ and $x, y \notin V$ so $(U \cup U_1) \cap (V \cap U_2) \cap \{u_2\}^c = \emptyset$

   $(U \cup U_1) \cup (V \cap U_2) \supset U_1 \cup U \cup (U_2 \cap (U \cup V)) \supset (U_1 \cup U_3 \cup U_2) \cap \{u_1, u_2\}^c \supset X \backslash \{u_1, u_2, x, y\}$ but $x, y \in U$, and $u_1 \in U_1$ so $((U \cup U_1) \cup (V \cap U_2)) \cap \{u_2\}^c = X \backslash \{u_2\}$

   $X \backslash \{u_2\}$ is connected and $x \in (U \cup U_1) \cap \{u_2\}^c \neq \emptyset$ so $(V \cap U_2) \cap \{u_2\}^c = \emptyset$

4. With the same previous step, we have $(V \cap U_1) \cap \{u_1\}^c = \emptyset$.

   So $V \cap \{u_1, u_2\}^c \subset (V \cap U_1 \cap \{u_1\}^c) \cup (V \cap U_2 \cap \{u_2\}^c) \cup (V \cap (U_3 \cup \{x, y\})) = \emptyset$. So, $X \backslash \{u_1, u_2\}$ is connected.

□

Theorem 2. *A n-flimsy space is infinite.*

Proof. see https://math.stackexchange.com/questions/2939445/flimsy-spaces-removing-any-n-points-results-in-disconnectedness for the proof of 'Babelfish' □

Theorem 3. *Let X a n-flimsy space. $\forall x \in X$, $\{x\}$ is either open or closed.*

Proof. We start with the case $n = 1$. X is connected but $X \backslash \{x\}$ is disconnected. It exists a nontrivial clopen set $Y \subset X \backslash \{x\}$, in particular $Y \neq \emptyset$ and $Y \cup \{x\} \neq X$. Since Y is open in $X \backslash \{x\}$, Y or $Y \cup \{x\}$ is open in X.

- if Y is open in X, by connectedness, Y is not closed in X. Since Y in closed in $X \backslash \{x\}$, $Y \cup \{x\}$ is closed in X. So, $\{x\} = (Y \cup \{x\}) \cap (X \backslash Y)$ is closed.
- if $Y \cup \{x\}$ is open in X, then Y is closed in X, and $\{x\} = (Y \cup \{x\}) \cap (X \backslash Y)$ is open.

By induction, we suppose the theorem to be true for $n \geq 1$, and we observe X a $(n+1)$-flimsy space and $x \in X$. X is infinite, so there is $y, z \in X$, $y \neq z$, such that $\{x\}$ is either open in $X \backslash \{y\}$ and $X \backslash \{z\}$ or closed in $X \backslash \{y\}$ and $X \backslash \{z\}$, because $X \backslash \{y\}$ and $X \backslash \{z\}$ are n-flimsy. We suppose we are in the open case (the closed case can be examined in the same way).

If $\{x\}$ is not open in X then $\{x, y\}$ and $\{x, z\}$ are open in X, so $\{x\} = \{x, y\} \cap \{x, z\}$ is open in X.

□

Lemma 1. *Let $x, y \in X$, two distinct points of a 2-flimsy space, and C one of the two connected components of $X \backslash \{x, y\}$. $C \cup \{x\}$ and $C \cup \{y\}$ are connected.*

Proof. By contradiction, we suppose $C \cup \{x\}$ is disconnected.

C and $\{x\}$ are connected, so they are the only connected components of $C \cup \{x\}$, so C is open and closed in $C \cup \{x\} \subset X\backslash\{y\}$. There is an open set U of $X\backslash\{y\}$ such that $C = U \cap (C \cup \{x\})$

Moreover, C is open and closed in $X\backslash\{x, y\}$, because it is one of its only two connected components, so C or $C \cup \{x\}$ is open in $X\backslash\{y\}$. But we know $C = U \cap (C \cup \{x\})$, so in every case, C is open in $X\backslash\{y\}$. The same shows C is closed in $X\backslash\{y\}$. C is not trivial so $X\backslash\{y\}$ is not connected: we have a contradiction. □

Let X a 2-flimsy space and x, y, z three distinct points of X. We denote $C(\{x, y\}, z \in)$ the connected component of $X\backslash\{x, y\}$ which contains z, and $C(\{x, y\}, z \notin)$ the other one. We also denote $C(\{x, x\}, z \in) = X\backslash\{x\}$ and $C(\{x, x\}, z \notin) = \emptyset$.

Some relations and basic properties:

$C(\{x, y\}, z \in) \cup C(\{x, y\}, z \notin) = X\backslash\{x, y\}$ and $C(\{x, y\}, z \in) \cap C(\{x, y\}, z \notin) = \emptyset$.

If $a \neq x, y$, then $a \in C(\{x, y\}, z \in) \Leftrightarrow C(\{x, y\}, z \in) = C(\{x, y\}, a \in)$ and $a \in C(\{x, y\}, z \notin) \Leftrightarrow C(\{x, y\}, z \notin) = C(\{x, y\}, a \in)$

Theorem 4. *If A is a connected subset of X, a 2-flimsy space, then A^c is also connected.*

Proof. If A is trivial, the result is obvious. We can choose some $a \in A$ and $\psi \in A^c$. We begin to prove the following equality.

$$A = \bigcap_{x \notin A} C(\{x, \psi\}, a \in)$$

Let $x \notin A$. $A \subset X\backslash\{x, \psi\}$, and $a \in A \cap C(\{x, \psi\}, a \in) \neq \emptyset$, so $A \subset C(\{x, \psi\}, a \in)$ by connectedness. Moreover, $x \notin C(\{x, \psi\}, a \in)$, then $x \notin \bigcap_{y \notin A} C(\{y, \psi\}, a \in)$. We obtain a new equality, with the complements.

$$A^c = \bigcup_{x \in A^c} C(\{x, \psi\}, a \notin) \cup \{x, \psi\}$$

Thanks to the previous lemma, we know the $C(\{x, \psi\}, a \notin) \cup \{x, \psi\}$ are connected, and they all contain ψ, so their union is also connected. □

Lemma 2. *Let $x, t, s \in X$, three distinct points of a 2-flimsy space.*

$$C(\{t, s\}, x \notin) = C(\{x, t\}, s \in) \cap C(\{x, s\}, t \in)$$

$$C(\{t, s\}, x \in) = C(\{x, t\}, s \notin) \cup C(\{x, s\}, t \notin) \cup \{x\}$$

Proof. First we can remark that

$$X\backslash\{t, s\} = [C(\{x, t\}, s \in) \cap C(\{x, s\}, t \in)] \cup [C(\{x, t\}, s \notin) \cup C(\{x, s\}, t \notin) \cup \{x\}]$$

$$\emptyset = [C(\{x, t\}, s \in) \cap C(\{x, s\}, t \in)] \cap [C(\{x, t\}, s \notin) \cup C(\{x, s\}, t \notin) \cup \{x\}]$$

so we only need to show these two sets are connected.

By the lemma 1, $C(\{x,t\}, s \notin) \cup \{x\}$ and $C(\{x,s\}, t \notin) \cup \{x\}$ are connected, so

$C(\{x,t\}, s \notin) \cup C(\{x,s\}, t \notin) \cup \{x\}$ is connected as their union.

$[C(\{x,t\}, s \in) \cap C(\{x,s\}, t \in)]^c = C(\{x,t\}, s \notin) \cup C(\{x,s\}, t \notin) \cup \{x,t,s\}$ is connected, because $C(\{x,t\}, s \notin) \cup \{x,t\}$ and $C(\{x,s\}, t \notin) \cup \{x,s\}$ are connected. The complement of a connected set is connected, which concludes the proof. □

Theorem 5. *There are no* 3*-flimsy spaces.*

Proof. Let X a 3-flimsy space and x,y,t,s some distinct points of X. $X\backslash\{y\}$ is 2-flimsy, so if C_1 is the connected component of $X\backslash\{y,x,t\}$ containing s and C_2 is the connected component of $X\backslash\{y,x,s\}$ containing t, then $D = C_1 \cap C_2$ is one of the two connected components of $X\backslash\{y,t,s\}$. Moreover, D is also one of the two connected components of $X\backslash\{x,t,s\}$, by using the lemma 2 in $X\backslash\{x\}$. $x,y,t,s \notin D$

So, D is open and closed in $X\backslash\{x,t,s\}$ and in $X\backslash\{y,t,s\}$. We have two open sets of X, U_x and U_y, and two closed sets of X, G_x and G_y, such that

$U_x \cap \{x,t,s\}^c = G_x \cap \{x,t,s\}^c = D$ and $U_y \cap \{y,t,s\}^c = G_y \cap \{y,t,s\}^c = D$, so $y \notin U_x, G_x$ and $x \notin U_y, G_y$

$U_x \cap U_y \cap \{t,s\}^c = U_x \cap \{y,t,s\}^c \cap U_y \cap \{x,t,s\}^c = D \cap D = D$ and also, $G_x \cap G_y \cap \{t,s\}^c = D$. Since $U_x \cap U_y$ is open in X and $G_x \cap G_y$ is closed in X, D is open and closed in $X\backslash\{t,s\}$. Moreover, D is not trivial because it is a connected component of $X\backslash\{x,t,s\}$. So $X\backslash\{t,s\}$ is not connected and X is not 3-flimsy.

□

Lemma 3. *If $s,t,u,v \in X$ are distinct such that $v \in C(\{s,t\}, u \notin)$, then $s \in C(\{u,v\}, t \notin)$.*

Proof.

$$C(\{u,v\}, t \notin) = C(\{u,t\}, v \in) \cap C(\{v,t\}, u \in)$$

$$v \in C(\{s,t\}, u \notin) = C(\{u,s\}, t \in) \cap C(\{u,t\}, s \in) \subset C(\{u,t\}, s \in)$$

So $C(\{u,t\}, v \in) = C(\{u,t\}, s \in)$ and $s \in C(\{u,t\}, v \in)$.

Moreover, $v \in C(\{s,t\}, u \notin)$ is the same thing as $u \in C(\{s,t\}, v \notin)$.

$$u \in C(\{s,t\}, v \notin) = C(\{v,s\}, t \in) \cap C(\{v,t\}, s \in) \subset C(\{v,t\}, s \in)$$

So $C(\{v,t\}, u \in) = C(\{v,t\}, s \in)$ and $s \in C(\{v,t\}, u \in)$.

Finally, $s \in C(\{u,t\}, v \in) \cap C(\{v,t\}, u \in) = C(\{u,v\}, t \notin)$. □

Theorem 6. *If X is a* 2*-flimsy space, then X is a Hausdorff space.*

Proof. Let $x \neq y$ two points of X. We choose some $a \in X$ distinct of x and y, and we take some $b \in C(\{x,y\}, a \notin)$. Then, we choose $\tilde{b} \in C(\{x,a\}, b \notin)$ and $\tilde{a} \in C(\{y,b\}, a \notin)$. Obviously, $x, y, a, b, \tilde{a}, \tilde{b}$ are distinct. We are going to show that $C(\{b,\tilde{b}\}, x \in) \cap C(\{a,\tilde{a}\}, y \in) = \emptyset$.

Because $\tilde{b} \in C(\{x,a\}, b \notin)$, by using the lemma, we have $a \in C(\{b,\tilde{b}\}, x \notin)$ and so

$$x \in C(\{b,\tilde{b}\}, x \in) = C(\{b,\tilde{b}\}, a \notin) \subset C(\{a,b\}, \tilde{b} \in).$$

This implies $C(\{a,b\}, \tilde{b} \in) = C(\{a,b\}, x \in)$ and $C(\{b,\tilde{b}\}, x \in) \subset C(\{a,b\}, x \in)$.

With the same method, we have $C(\{a,\tilde{a}\}, y \in) \subset C(\{a,b\}, y \in)$. But since $b \in C(\{x,y\}, a \notin)$, by using the lemma, we have $y \in C(\{a,b\}, x \notin)$, so $C(\{a,b\}, y \in) = C(\{a,b\}, x \notin)$ and we conclude

$$C(\{b,\tilde{b}\}, x \in) \cap C(\{a,\tilde{a}\}, y \in) \subset C(\{a,b\}, x \in) \cap C(\{a,b\}, x \notin) = \emptyset$$

$C(\{b,\tilde{b}\}, x \in)$ is open in $X \backslash \{b,\tilde{b}\}$, so there is an open set U of X such that $U \cap \{b,\tilde{b}\}^c = C(\{b,\tilde{b}\}, x \in)$. We have also V an open set such that $V \cap \{a,\tilde{a}\}^c = C(\{a,\tilde{a}\}, y \in)$. In particular, $x \in U$ and $y \in V$.

$$U \cap V \subset (C(\{b,\tilde{b}\}, x \in) \cup \{b,\tilde{b}\}) \cap (C(\{a,\tilde{a}\}, y \in) \cup \{a,\tilde{a}\})$$

We need to show $a, \tilde{a} \notin C(\{b,\tilde{b}\}, x \in)$, ($b, \tilde{b} \notin C(\{a,\tilde{a}\}, y \in)$ will follow by symmetry), to conclude $U \cap V = \emptyset$. However, we have already shown $a \in C(\{b,\tilde{b}\}, x \notin)$. Also recall that $C(\{b,\tilde{b}\}, x \in) \subset C(\{a,b\}, x \in)$.

$$\tilde{a} \in C(\{y,b\}, a \notin) \subset C(\{a,b\}, y \in) = C(\{a,b\}, x \notin)$$

So, $\tilde{a} \notin C(\{a,b\}, x \in)$ and $\tilde{a} \notin C(\{b,\tilde{b}\}, x \in)$. □

Theorem 7. *Let X a 2-flimsy space, $x, y \in X$, and C a connected component of $X \backslash \{x,y\}$. Then, C is open in X and $C \cup \{x,y\}$ is closed in X. Moreover, if $x \neq y$, then C is the interior of $C \cup \{x,y\}$, and $C \cup \{x,y\}$ is the closure of C.*

The connected components of X without any two points form a base of a coarser topology on X. With this topology, X is still a 2-flimsy space and the connected components of X without any two points remain the same.

Proof. If $x = y$, $C = X \backslash \{x\}$ is open because X is Hausdorff.

If $x \neq y$, C is closed in $X \backslash \{x,y\}$, and since $\{x,y\}$ is closed in X (Hausdorff), $C \cup \{x,y\}$ is closed in X. By the connectedness of X, $X \backslash \{x\}$, and $X \backslash \{y\}$, $C \cup \{x,y\}$, $C \cup \{y\}$, and $C \cup \{x\}$ are not open in X. However, C is open in $X \backslash \{x,y\}$, so C is open in X. The connectedness implies $C \cup \{y\}$ and $C \cup \{x\}$ are not closed in X, and the identities on the closure and the interior follow.

It is easy to verify that the intersection of two sets of the form $C(\{x,y\}, t \in)$ is either empty, either of the form $C(\{x,y\}, t \in)$, either an union of two sets of the form $C(\{x,y\}, t \in)$. In the topology generated by the $C(\{x,y\}, t \in)$, all the $C(\{x,y\}, t \in)$ are open and all the $C(\{x,y\}, t \in) \cup \{x,y\}$ are closed. So for any $x \neq y$, $X \backslash \{x,y\}$ is not connected.

□

1 Omega flimsy

1.1 Small results

In all this section, X is an omega-flimsy topological space.

If T is a countable infinite subset of X, then $X\backslash T$ is not connected. Thus, there are two open sets U and V of X, such that $U \cap V \subset T$, $(U \cup V)^c \subset T$, $U \cap T^c \neq \emptyset$, and $V \cap T^c \neq \emptyset$. But, it implies that $(U \cap V) \cup (U \cup V)^c$ is countable while $X\backslash[(U \cap V) \cup (U \cup V)^c]$ is disconnected. So, $(U \cap V) \cup (U \cup V)^c$ is infinite. Finally, by observing $(U \cap V)$ is open and $(U \cup V)^c$ is closed, we conclude

Lemma 4. *If T is a countable infinite subset of X, then there is an infinite $S \subset T$ which is open or closed.*

Lemma 5. *Except for a finite number of points, for all $x \in X$, $\{x\}$ is either open or closed.*

Proof. Let us think by contradiction, and let us suppose $T = \{x_n,\ n \in \mathbb{N}\}$ is infinite countable subset of X such that $\forall n \in \mathbb{N}$, $\{x_n\}$ is nor open nor closed. Either, each infinite subset of T contains a closed infinite subset, either T has a subset T' which does not contain any closed infinite subset. So, each infinite subset of T' contains an open infinite subset. Without loss of generality, we can suppose each infinite subset of T contains a closed infinite subset (the case with open is similar). We define $T_n = \{x_{p_n^k},\ k \geq 1\}$ where p_n is the n-th prime number. There are disjoint subsets of T.

For all $n \geq 1$, there is $S_n \subset T_n$, a closed infinite subset. We can construct a strictly increasing sequence $(a_n)_{n\geq 1}$ of indexes, such that $x_{a_n} \in S_n$ for all $n \geq 1$. But there is $S \subset \{x_{a_n},\ n \geq 1\}$, which is closed and infinite. For an infinity of n, $S \cap S_n = \{x_n\}$ is closed, which is a contradiction.

□

The same argument can be adapted to show the following result.

Lemma 6. *Let T be a countable infinite subset of X such that $\forall x \in T,\ \{x\}$ is closed. There is an infinite closed set $S \subset T$ with empty interior.*

If T does not contain any infinite closed subset, then each infinite subset of T contains an open infinite subset, and there is $x \in T$ such that $\{x\}$ is open, which contradicts the connectedness of X. If S is an infinite closed subset of T, then $S\backslash\mathring{S} = \partial S$ satisfies all the desired properties. In particular, it is infinite because X is omega-flimsy.

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If $S \subset T$ and if A is a clopen set of $X\backslash S$. $B = A \cap T^c$ is a clopen set of $X\backslash T$ such that $B \subset A \subset B \cup T$. Now, we place ourselves in X. There is an open set U and a closed set F such that $U \cap S^c = F \cap S^c = A$ and $U \cap T^c = F \cap T^c = B$. But $B = U \cap T^c$ is also open. $B \cap F^c \subset T$ and since $B \cap F^c$ is open, $B \cap F^c = \emptyset$, or in other words $B \subset F$, because T has an empty interior. In the same way, $U \subset \overline{B}$. So, $B \subset U \subset \mathring{\overline{B}} \subset \overline{B} \subset F \subset B \cup T$ and $\overline{B}\backslash\mathring{\overline{B}} \subset S$.

Conversely, if B is a clopen set of $X\backslash T$ such that $\overline{B}\backslash\overset{\circ}{\overline{B}} \subset S$, then $A = \overline{B} \cap S^c = \overset{\circ}{\overline{B}} \cap S^c$ is a clopen set of $X\backslash S$ such that $B \subset A \subset B \cup T$.

Moreover, if $B = \emptyset$, then $U \subset T$, and since U is open, $U = \emptyset = A$. In the same way, if $B = X\backslash T$ then $A = X\backslash S$. So, B is trivial if and only if A is trivial.

Proposition 1. *If T is a countable infinite closed set with empty interior, then*

- *for any B clopen set of $X\backslash T$, there is A a clopen set of $X\backslash S$ such that $B \subset A \subset B \cup T$ if and only if $\overline{B}\backslash\overset{\circ}{\overline{B}} \subset S$.*
- *$X\backslash S$ is disconnected if and only if there is B a non-trivial clopen set of $X\backslash T$ such that $\overline{B}\backslash\overset{\circ}{\overline{B}} \subset S$.*

We denote $S(B) = \overline{B}\backslash\overset{\circ}{\overline{B}}$.

Definition. Let T be a countable infinite set. A subset $\mathcal{A}$ of $\mathcal{P}(T)$ is said to be mirific in T when:

- for all $A \in \mathcal{A}$, A is infinite.
- for all infinite $S \subset T$, there is $A \in \mathcal{A}$ such that $A \subset T$

If T is a countable infinite closed set with empty interior, then $\{S(B),\ B$ non-trivial clopen set of $X\backslash T\}$ is mirific in T.

Lemma 7. *A mirific set can not be countable.*

Proof. $\mathcal{A} = \{A_n,\ n \geq 1\}$ mirific in $\mathbb{N}$. Choose in each A_n a x_n distinct of the previous and such that $x_n \geq 2^n$ to find a contradiction □

1.2 The connected subsets

The property of being omega-flimsy implies much more disconnectedness than it appears at first. We place ourselves in X, an omega-flimsy space.

Theorem 8. *The connected subsets of an omega-flimsy space are either finite or cofinite.*

Definition. A subset T of X is said to be alike either if T is closed and $\overset{\circ}{T} = \emptyset$ or if $\forall x \in T,\ \{x\}$ is open.

By connectedness of X, if T is alike and open (resp. closed), than its only closed (resp. open) subset is $\emptyset$. We have already proven that if T is infinite, it has an infinite alike subset.

Lemma 8. *If T is a countable infinite alike subset of X, then $X\backslash T$ has an infinity of connected components.*

Proof. We assume that T is closed (interchanging the words 'closed' and 'open' gives the case where T is open).

We begin by observing that if A is a non-trivial clopen subset of $X\backslash T$ then

$$\mathcal{F}_A = \{S \subset T,\ \exists B \text{ clopen of } X\backslash S \text{ such that } B \cap T^c = A\}$$

is a filter on T. The facts that $T \in \mathcal{F}_A$ and ($R \subset S$ and $R \in \mathcal{F}_A \implies S \in \mathcal{F}_A$) are obvious. Moreover, $\emptyset \notin \mathcal{F}_A$ because X is connected. Let $S_1, S_2 \in \mathcal{F}_A$ and let us prove that $S_1 \cap S_2 \in \mathcal{F}_A$. We have U_1, U_2 some open sets in X and F_1, F_2 some closed sets in X such that $U_1 \cap S_1^c = F_1 \cap S_1^c$, $U_2 \cap S_2^c = F_2 \cap S_2^c$, and $F_1 \cap T^c = F_2 \cap T^c = A$. The set $(U_1 \cup U_1) \cap (F_1 \cap F_2)^c$ is open and included in T, so it is empty.

$$(F_1 \cap F_2)\backslash(U_1 \cup U_2) = (F_1\backslash U_1) \cap (F_2\backslash U_2) \subset S_1 \cap S_2$$

So, $(F_1 \cap F_2) \cap (S_1 \cap S_2)^c = (U_1 \cup U_2) \cap (S_1 \cap S_2)^c$ and $(F_1 \cap F_2) \cap T^c = A$, and $S_1 \cap S_2 \in \mathcal{F}_A$.

Now, by contradiction, we suppose that $X\backslash T$ has an finite number of connected components. Then, it also has a finite number of non-trivial clopen subsets (more precisely, if it has n connected components than it has $2^n - 2$ non-trivial clopen subsets). By considering a sequence of disjoint infinite subsets of T, one of them is not in any of the filters associated to those clopen subsets of $X\backslash T$. Indeed, if it was not the case, one of the filters would contain two disjoint subsets and so would contain $\emptyset$ which is impossible. We choose S such subset of T. If B is a clopen subset of $X\backslash S$, then $B \cap T^c$ is a clopen subset of $X\backslash T$ but is not a non-trivial clopen subset of $X\backslash T$, so $B \cap T^c = \emptyset$ or T^c. Without loss of generality (may by taking $(X\backslash S)\backslash B$), we can assume that $B \subset T$. Since B is open in $X\backslash S$, there is an open subset U such that $U \cap S^c = B$, so $U \subset T$ and $U = \emptyset$, thus $B = \emptyset$. The set $X\backslash S$ is connected which is a contradiction with the fact that X is omega-flimsy. □

The following lemma is a general result which is not too difficult to prove.

Lemma 9. *Let Y be a topological space with an infinity of connected components and let C be a connected subset of Y. There exists a sequence $(A_n)_{n\geq 0}$ of disjoint non-empty clopen subsets, all disjoint from C.*

Proof. pRoOf Is LeFt FoR tHe ReAdEr □

Now, we can begin to look at the connected subsets of X.

Proposition 2. *If C a connected subset and if T is a countable infinite alike subset disjoint from C, then $C \cup T$ is not connected. Typically, if C is a connected subset then $\overline{C}\backslash C$ is finite.*

Proof. Let us suppose by contradiction that $C \cup T$ is connected. There exists a sequence $(A_n)_{n\geq 0}$ of disjoint non-empty clopen sets of $X\backslash T$, all disjoint from C (because C is connected and T is alike).

First case: We suppose for all $n \geq 0$, there exists $a_n \in A_n$ such that $\{a_n\}$ is not open in X. In this case, we set $S = \{a_n,\ n \geq 0\}$.

Let P and Q two open sets of X such that $P \cap Q \subset S$ and $X \backslash S \subset P \cup Q$. Then, $P \cap Q \cap (C \cup T) = \emptyset$ and $(P \cup Q) \cap (C \cup T) = C \cup T$ because S is disjoint from $C \cup T$. Since $C \cup T$ is connected, we can assume without loss of generality that $C \cup T \subset P$ and $(C \cup T) \cap Q = \emptyset$. We are going to show that $Q = \emptyset$ which will contradict the disconnectedness of $X \backslash S$.

For all $n \geq 0$, we have U_n an open set and F_n a closed set such that $U_n \cap T^c = F_n \cap T^c = A_n$. Let us compare $U_n \cap Q$ and $F_n \cap P^c$ which are respectively open and closed.

$$\begin{aligned} (U_n \cap Q)\Delta(F_n \cap P^c) &\subset [(U_n \Delta F_n) \cap (Q \cup P^c)] \cup [(Q \Delta P^c) \cap (U_n \cup F_n)] \\ &\subset [T \cap (Q \cup P^c)] \cup [S \cap (A_n \cup T)] \\ &\subset S \cap A_n \\ &\subset \{a_n\} \end{aligned}$$

Because $X \backslash \{a_n\}$ is connected and $T \cap U_n \cap Q = \emptyset$, it implies that $U_n \cap Q \subset \{a_n\}$. Moreover, we know that $\{a_n\}$ is not open, so $U_n \cap Q = \emptyset$. In particular, $a_n \notin Q$ (because $a_n \in A_n \subset U_n$), so $Q \cap S = \emptyset$ and $P \cap Q = \emptyset$.

To conclude, we see that $P \cup \bigcup_{n \geq 0} U_n$ and Q constitute an open partition of X, and so Q is trivial. Indeed, we have already $\left(P \cup \bigcup_{n \geq 0} U_n\right) \cap Q = (P \cap Q) \cup \bigcup_{n \geq 0} U_n \cap Q = \emptyset$. Plus, $S \subset \bigcup_{n \geq 0} U_n$ so $\left(P \cup \bigcup_{n \geq 0} U_n\right) \cup Q = X$.

Second case: We assume $\forall n \geq 0, \forall a_n \in A_n,\ \{a_n\}$ is open in X. We choose for each $n \geq 0$ some $a_n \in A_n$ and we set $S = \{a_n,\ n \geq 0\}$. S is an open and alike subset of X. We remark $\{a_n\}$ is also closed in $X \backslash T$ because $\{a_n\} = A_n \backslash \bigcup_{\substack{a \in A_n \\ a \neq a_n}} \{a\}$. The connected set $\overline{\{a_n\}}$ is included in $\{a_n\} \cup T$ and is not disjoint from T, otherwise $\{a_n\}$ would be a clopen set of X. We deduce that $C \cup T \cup \{a_n\} = C \cup T \cup \overline{\{a_n\}}$ is connected, and so is $C \cup T \cup S$.

Now, if B is a closed set of $X \backslash S$ which contains only singletons that are open in X, then B is open in X as an union of open singletons. However, since S is open in X, B is also closed in X. By connectedness of X, $B = \emptyset$. Finally, with $C \cup T$ and S, we are under the assumptions of the first case. □

Corollary 1. *$\forall x \in X,\ \{x\}$ is not open. Except for a finite number of points, for all $x \in X$, $\{x\}$ is closed.*

Proof. If $\{x\}$ is open, then $\overline{\{x\}}$ is infinite because X is omega-flimsy, but can not be infinite because $\{x\}$ is connected. □

Proposition 3. *If C is an infinite and co-infinite connected subset of X, then $X \backslash C$ has an infinity of connected components.*

Proof. By contradiction, we write $X = \bigsqcup_{i=1}^{n} C_i$ where $n \in \mathbb{N}$, the C_i are connected, and $C = C_1$. It is not difficult to see

$$\overline{C}\backslash\mathring{C} \subset \bigcup_{i=1}^{n} \overline{C_i}\backslash C_i.$$

Hence, the set $\partial C = \overline{C}\backslash\mathring{C}$ is finite and $\mathring{C}$ is a clopen subset of $X\backslash\partial C$ which is connected. So, $\mathring{C} = \emptyset$ or $\mathring{C} = X\backslash\partial C$. In the first case, $C \subset \partial C$ is finite. In the second case, $C \supset X\backslash\partial C$ is cofinite. □

We are finally able to prove the theorem.

Proof. Let C be an infinite and co-infinite connected subset of X. According to the previous proposition, there exists a sequence $(A_n)_{n\geq 0}$ of disjoint non-empty clopen sets in $X\backslash C$. For all $n \geq 0$, we choose $a_n \in A_n$, and we automatically know such that $\{a_n\}$ is not open in X (we even can ask closed in X). We set $S = \{a_n,\ n \geq 0\}$. The following of the proof is similar to the first case of the first proposition.

Let P and Q two open sets of X such that $P \cap Q \subset S$ and $X\backslash S \subset P \cup Q$. Then, $P \cap Q \cap C = \emptyset$ and $(P \cup Q) \cap C = C$ because S is disjoint from C. Since C is connected, we can assume without loss of generality that $C \subset P$ and $C \cap Q = \emptyset$. We are going to show that $Q \subset S$ which will contradict the disconnectedness of $X\backslash S$.

For all $n \geq 0$, we have U_n an open set and F_n a closed set such that $U_n \cap C^c = F_n \cap C^c = A_n$. Let us compare $U_n \cap Q$ and $F_n \cap P^c$ which are respectively open and closed.

$$\begin{aligned}(U_n \cap Q)\Delta(F_n \cap P^c) &\subset [(U_n\Delta F_n) \cap (Q \cup P^c)] \cup [(Q\Delta P^c) \cap (U_n \cup F_n)]\\ &\subset [C \cap (Q \cup P^c)] \cup [S \cap (A_n \cup C)]\\ &\subset S \cap A_n\\ &\subset \{a_n\}\end{aligned}$$

Because $X\backslash\{a_n\}$ is connected and $C \cap U_n \cap Q = \emptyset$, it implies that $U_n \cap Q \subset \{a_n\}$. Moreover, we know that $\{a_n\}$ is not open, so $U_n \cap Q = \emptyset$. In particular, $a_n \notin Q$ (because $a_n \in A_n \subset U_n$), so $Q \cap S = \emptyset$ and $P \cap Q = \emptyset$.

To conclude, we see that $P \cup \bigcup_{n\geq 0} U_n$ and Q constitute an open partition of X, and so Q is trivial. Indeed, we have already $\left(P \cup \bigcup_{n\geq 0} U_n\right) \cap Q = (P \cap Q) \cup \bigcup_{n\geq 0} U_n \cap Q = \emptyset$. Plus, $S \subset \bigcup_{n\geq 0} U_n$ so $\left(P \cup \bigcup_{n\geq 0} U_n\right) \cup Q = X$. □

Corollary 2. *There exists an omega-flimsy space if and only if there exists an uncountable topological space in which the non-degenerate connected sets are exactly the cofinite sets.*

Proof. By removing from an omega-flimsy space the finite number of singletons which are not closed, we obtain a T1 omega-flimsy space. The finite connected subsets of a T1 space are degenerate. The cofinite subsets are connected by definition. An omega-flimsy space is uncountable because $\emptyset$ is always connected. □