Study of a division-like property

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Abstract: We study a weak divisibility property for noncommutative rings: a nontrivial ring is *fadelian* if for all nonzero a and x there exist b, c such that $x = ab + ca$. We prove properties of fadelian rings and construct examples thereof which are not division rings, as well as non-Noetherian and non-Ore examples.

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All rings considered in this article are nontrivial, associative, unital, and not assumed to be commutative.

1. Introduction

Various notions of weak inversibility in noncommutative rings have been considered, notably (strong) von Neumann regularity [Neu36] and unit regularity. We introduce a class of rings satisfying another weak form of divisibility. They are the *fadelian* and *weakly fadelian* rings:[1]

Definition 1.1. A nontrivial ring R is:

- *fadelian* if for any $x \in R$ and any nonzero $a \in R$, there exist $b, c \in R$ such that $x = ab + ca$;
- *weakly fadelian* if for any nonzero $a \in R$, there exist $b, c \in R$ such that $1 = ab + ca$.

The properties of Definition 1.1 have already appeared in [LJL09, Theorem 3.2, (iv) and (vii)] as equivalent characterizations of V*-domains* in the case of atomic left principal ideal domains. In that article, the left-right symmetry of the fadelianity property is used to prove that left V-domains and right V-domains coincide. In this work, we study this property outside of this context, to understand how it constrains the structure of a general ring. The starting point is the natural sequence of implications:

$$R \text{ is a division ring} \Rightarrow R \text{ is fadelian} \Rightarrow R \text{ is weakly fadelian.}$$

Asking whether these implications are equivalences, we try to construct counterexamples when they are not. Our results are summed up in the following paragraph, which also serves as an outline:

- In Section 2, we prove that weakly fadelian rings are simple domains (Proposition 2.1, Theorem 2.2), and that weakly fadelian Ore rings are fadelian (Theorem 2.6).

[1] We named these rings after our colleague Assil Fadle, thanks to whom we got interested in this topic. To our knowledge, this property has not previously been studied or given a name.

- In Section 3, we give necessary and sufficient conditions for a differential polynomial ring to be fadelian (Proposition 3.2.1 and Theorem 3.3.1). The sufficient condition is satisfied if the base ring is a differentially closed field — this gives first examples of fadelian rings which are not division rings. In Theorem 3.4.1 and Theorem 3.5.2, we transform this example into a countable non-Noetherian fadelian ring using ultraproducts and model-theoretic arguments.
- In Section 4, we construct a non-Ore fadelian ring (Theorem 4.4, Corollary 4.5), hence proving that the implication of Theorem 2.6 is not an equivalence. The main tool is the fact that rings of Laurent series over fadelian rings are themselves fadelian (Theorem 4.3).

The main remaining question is whether there is a weakly fadelian ring which is not fadelian. It is also worth investigating whether fadelian rings necessarily have the invariant basis number property, which is weaker than the Ore condition. The authors thank André Leroy and Maxime Ramzi, as well as the anonymous referee, for their interest and their suggestions.

Remark 1.2. The equation $x = ab + ca$ (solving for b, c when x and a are fixed) defining fadelian rings is visually similar to the "metro equation'' $ax - xb = d$ (solving for x when a, b, d are fixed) [LL04] and to the exchange equation" $xa - fx = 1$ (where a is fixed and f is idempotent) [KLN17]. However, the roles of parameters and indeterminates are modified (in particular, our equation has two free variables). Furthermore, our focus lies not on the equation itself but rather on understanding how its solvability impacts the structure of a ring — just like the theory of division rings studies the consequences of the solvability of the equation $1 = ab$ for all $a \neq 0$.

2. Properties of weakly fadelian rings

In this section, we prove properties of a fixed weakly fadelian ring R.[2] A first observation is that if R is commutative, it is a field. More generally:

Proposition 2.1. The ring R is simple.

Proof. Let I be a nonzero two-sided ideal of R. Let $a \in I \setminus \{0\}$. Since R is weakly fadelian, there are $b, c \in R$ such that $1 = ab + ca$. We obtain $1 \in I$ and finally $I = R$. □

We now prove the following result, which is the first non-trivial observation concerning weakly fadelian rings:

Theorem 2.2. The ring R is a domain.

The proof uses two lemmas:

Lemma 2.3. Assume $x, y \in R$ satisfy $xy = yx = 0$. Then $x^2 = 0$ or $y^2 = 0$.

[2] Proofs of Theorem 2.2 and Theorem 2.6 have been formalized in the Lean 4.2 proof assistant. The code for this formalization is available at https://beranger-seguin.fr/dmi/fadelian/fad_rings4.lean.

Proof. If $x = 0$, this is immediate. Otherwise, use weak fadelianity to write $1 = xb + cx$ for some $b, c \in R$. Then:

$$y^2 = y \cdot 1 \cdot y = y(xb + cx)y = yxby + ycxy = (yx)by + yc(xy) = 0.$$

$\square$

Lemma 2.4. Assume $x \in R$ satisfies $x^2 = 0$. Then $x = 0$.

Proof. Assume by contradiction that x is nonzero. Write:

$$1 = xb + cx \tag{1}$$

for some $b, c \in R$. Notice that:

$$cx = cx \cdot 1 = cx(xb + cx) = c(x^2)b + (cx)^2 = (cx)^2.$$

Similarly, we have $xb = (xb)^2$. Since $xb = 1 - cx$, we know that xb and cx commute. Hence:

$$(xb)(cx) = (cx)(xb) = c(x^2)b = 0.$$

By Lemma 2.3, we have $(xb)^2 = 0$ or $(cx)^2 = 0$, and thus $xb = 0$ or $cx = 0$. Assume for example that $xb = 0$. Equation 1 becomes $1 = cx$, so x is invertible, which contradicts $x^2 = 0$. $\square$

We finally prove Theorem 2.2:

Proof of Theorem 2.2. Let $x, y \in R$ such that $xy = 0$. Then $(yx)^2 = y(xy)x = 0$, which implies $yx = 0$ by Lemma 2.4. By Lemma 2.3, we deduce from $xy = yx = 0$ that either $x^2 = 0$ or $y^2 = 0$. Applying Lemma 2.4 again, we see that either x or y is zero. $\square$

The Ore condition is a well-studied condition, equivalent to the existence of a ring of fractions unique up to isomorphism [Ore31]. It is weaker than Noetherianity ([GR04, Corollary 6.7, Gol58, Theorem 1]). We recall the definition:

Definition 2.5. The domain R is right (resp. left) Ore if any two nonzero right (resp. left) ideals have a nonzero intersection.

The Ore condition interacts with fadelianity in the following way:

Theorem 2.6. If the weakly fadelian ring R is right Ore, then it is fadelian.

Proof. Assume R is right Ore. Let $x, a \in R \setminus \{0\}$. Since R is right Ore, there are nonzero elements $b, c \in R$ such that $ab = xc$. By Theorem 2.2, we have $ca \neq 0$. Since R is weakly fadelian, there are elements $k, k' \in R$ such that $1 = cak + k'ca$. Finally:

$$x = x \cdot 1 = xcak + xk'ca = abak + xk'ca \in aR + Ra.$$

This proves that R is fadelian. $\square$

In Corollary 4.5, we will see that Theorem 2.6 is not an equivalence.

3. Fadelianity and differential polynomial rings

In this section, we give conditions (both necessary and sufficient) for a differential polynomial ring to be fadelian.

3.1. Differential polynomial rings

Definition 3.1.1. Let k be a commutative field and R be a central k-algebra. A *derivation* of R is a nonzero k-linear map $\delta : R \to R$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. We say that (R, δ) is a differential algebra.

We fix a differential algebra (R, δ). Note that $\delta(1) = \delta(1 \cdot 1) = 2\delta(1)$ and hence $\delta(1) = 0$. In particular, δ is a non-invertible endomorphism. We now define differential polynomial rings:

Definition 3.1.2. For every $x \in R$, let $\tilde{x} \in \mathrm{End}(R)$ be the left multiplication endomorphism mapping an element $y \in R$ to the element xy. We denote by $R[\delta]$ the subalgebra of $\mathrm{End}(R)$ generated by the derivation δ and the endomorphisms $\tilde{x}$ for $x \in R$.

An introduction to differential polynomial rings (under the name "formal differential operator rings") is found in [GR04, Chapter 2]. These rings are particular cases of *Ore extensions.*

Remark 3.1.3. The ring $R[\delta]$ contains the nonzero non-invertible endomorphism δ, and is therefore never a division ring.

The map $x \mapsto \tilde{x}$ is an embedding of R into $R[\delta]$. We see R as a subalgebra of $R[\delta]$, i.e. we identify elements $x \in R$ with their associated left multiplication endomorphism $\tilde{x} \in R[\delta]$. To avoid any confusion between $\delta x = \delta \circ \tilde{x} \in R[\delta]$ and $\delta(x) \in R$, we now solely use the notation x' when evaluating δ at an element $x \in R$. By x^n, we mean $\delta^n(x)$.

Remark 3.1.4. Let $x \in R$. The identity $(xy)' = x'y + xy'$ for $y \in R$ gives the equality $\delta x = x' + x\delta$ of endomorphisms in $R[\delta]$, which can be rewritten as $[\delta, x] = x'$, where $[a, b]$ stands for the commutator $ab - ba$. In particular, the ring $R[\delta]$ is noncommutative: no $x \in R \setminus \ker(\delta)$ commutes with δ.

3.2. Necessary conditions for $R[\delta]$ to be fadelian

We prove Proposition 3.2.1, which gives necessary conditions on the differential algebra (R, δ) for $R[\delta]$ to be weakly fadelian. A consequence is that this construction cannot give examples of weakly fadelian rings which are not fadelian.

Proposition 3.2.1. Let (R, δ) be a differential algebra such that $R[\delta]$ is weakly fadelian. Then:

- R is a division ring;
- $R[\delta]$ is fadelian;
- every nonzero element of $R[\delta]$ is surjective as an endomorphism of R.

Note that the condition that every nonzero endomorphism of $R[\delta]$ is surjective means that R contains solutions to all nontrivial linear "differential equations".

Proof of Proposition 3.2.1. Let a be a nonzero element of R. In $R[\delta]$, write by weak fadelianity:

$$1 = ba\delta + a\delta c$$

with $b, c \in R[\delta]$. Evaluate this equality of endomorphisms at $1 \in R$:

$$1 = b(a \cdot 1') + a \cdot (c(1))' = a \cdot (c(1))'.$$

The element $(c(1))' \in R$ is an inverse of a in R. This shows that R is a division ring. In particular, R is right Noetherian. By [GR04, Theorem 2.6], $R[\delta]$ is right Noetherian too. In particular, $R[\delta]$ is Ore by [Gol58, Theorem 1]. Finally, Theorem 2.6 implies that $R[\delta]$ is fadelian.

Now, consider a nonzero element $u \in R[\delta]$. Let $a \in R$. By fadelianity of $R[\delta]$, write:

$$a = d(u\delta) + (u\delta)e$$

with $d, e \in R[\delta]$. Evaluate this equality of endomorphisms at $1 \in R$ to obtain:

$$a = d(u(1')) + u\big(e(1)'\big) = u\big(e(1)'\big) \in \mathrm{Im}(u).$$

This proves the surjectivity of u. □

3.3. A criterion for the fadelianity of $R[\delta]$

The main theorem of this subsection is the following:

Theorem 3.3.1. Let (R, δ) be a differential algebra with R commutative. The ring $R[\delta]$ is weakly fadelian if and only if R is a field and every nonzero element of $R[\delta]$ is surjective as an endomorphism of R.

Theorem 3.3.1 gives the first interesting examples of fadelian rings:

Corollary 3.3.2. If (k, δ) is a differentially closed field with $\delta \neq 0$, the ring $k[\delta]$ is fadelian and is not a division ring.

Differentially closed fields and the existence of differential closures for fields were considered in [Rob59]. See [Mar02, Theorem 6.4.10] for a modern approach, and for details about the model theory of differential fields.

The constructions of this subsection are not new: the examples of fadelian rings constructed using Corollary 3.3.2 are similar to the examples of V-domains constructed by Cozzens in [CF75, Theorem 5.21], which are fadelian by [LJL09, Theorem 3.2]. However, our proofs are direct and avoid the theory of V-rings and their modules completely.

The direct implication in Theorem 3.3.1 is a special case of Proposition 3.2.1. Therefore, we focus on proving the converse. **We now assume that R is a differential (commutative) field and that every nonzero endomorphism in $R[\delta]$ is surjective**, and our goal is to prove that $R[\delta]$ is fadelian.

3.3.1. Euclidean division

Lemma 3.3.1.1. Every element $u \in R[\delta]$ admits a unique representation as a sum:

$$u = \sum_{i \geq 0} a_i \delta^i$$

where all but finitely many elements $a_i \in R$ are zero.

Proof. The existence follows the fact that one can use the equality $\delta a = a\delta + a'$ repeatedly to make sure that all occurrences of δ are on the right side of each term of u. Let us prove the uniqueness. Assume that, for some sequences (a_i) and (b_i) of elements of R (all zero but finitely many), we have $\sum a_i \delta^i = \sum b_i \delta^i$. By hypothesis, $\delta \in R[\delta] \setminus \{0\}$ is surjective, so we fix elements $x_n \in R$ such that $(x_n)^{(n)} = 1$. We show inductively that $a_i = b_i$ for all $i \in \mathbb{Z}_{\geq 0}$. First, evaluate the equality $\sum a_i \delta^i = \sum b_i \delta^i$ at 0 to obtain $a_0 = b_0$. Now assume $a_i = b_i$ for all $i < k$. We have:

$$\left(\sum a_i \delta^i\right)(x_k) - \sum_{i=0}^{k-1} a_i x_k^{(i)} = \sum_{i \geq k} a_i x_k^{(i)} = a_k$$

but also:

$$\left(\sum a_i \delta^i\right)(x_k) - \sum_{i=0}^{k-1} a_i x_k^{(i)} = \left(\sum b_i \delta^i\right)(x_k) - \sum_{i=0}^{k-1} b_i x_k^{(i)} = \sum_{i \geq k} b_i x_k^{(i)} = b_k.$$

So $a_k = b_k$. This concludes the proof. □

Definition 3.3.1.2. Let $u \in R[\delta]$, decomposed as $u = \sum a_i \delta^i$ as in Lemma 3.3.1.1. If $u \neq 0$, the *degree* $\theta(u)$ of u is the largest i such that $a_i \neq 0$. Moreover, we let $\theta(0) = -\infty$.

Consider the skew polynomial ring $R[x;\delta]$ (cf. [GR04, Chapter 2]). The uniqueness part of Lemma 3.3.1.1 means that the surjective map

$$\begin{cases} R[x;\delta] \to R[\delta] \\ x \qquad\ \ \mapsto \delta \end{cases}$$

is injective, i.e. that δ is not algebraic (under the assumption that it is surjective). In other words, $R[\delta]$ is isomorphic to the skew polynomial ring $R[x; \mathrm{id}, \delta]$. Since $R[x;\delta]$ is right and left Euclidean [Ore33, Section I.2], we directly obtain:

Lemma 3.3.1.3. The map $\theta : R[\delta] \to \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ is a Euclidean valuation for which $R[\delta]$ admits left and right Euclidean division. In particular, $R[\delta]$ is a (left and right) principal ideal domain.

For instance, left Euclidean division means that if $x, y \in R[\delta]$ with $y \neq 0$, there exist $q, r \in R[\delta]$ such that $x = qy + r$ and $\theta(r) < \theta(y)$. This can also be proved directly by induction on $\theta(x)$.

3.3.2. Diagonally dominant systems in differential fields

Lemma 3.3.2.1. Consider a system of equations of the form:

$$\forall j \in \{0, ..., r\}, \sum_{i=0}^{r} P_{i,j}(b_i) = x_j$$

in the indeterminates $b_0, ..., b_r$, where $x_j \in R$ and $P_{i,j} \in R[\delta]$. Assume moreover that:

$$\forall j \in \{0, ..., r\}, \theta(P_{j,j}) > \max_{i \neq j} \theta(P_{i,j}).$$

Then, the system admits a solution $(b_0, ..., b_r) \in R^{r+1}$.

Proof. In Lemma 3.3.1.3, we have established that $R[\delta]$ is a left principal ideal domain. By the noncommutative version of the Smith normal form [Jac43, Chapter 3, Theorem 16], the square matrix

$$P = (P_{i,j})_{0 \leq i,j \leq r}$$

with coefficients in $R[\delta]$ associated to our system is equivalent to a diagonal matrix $\tilde{P} = \mathrm{Diag}\left(\widetilde{P_0}, ..., \widetilde{P_r}\right)$. More precisely, there are invertible square matrices A, B of size $r+1$ with coefficients in $R[\delta]$ such that $\tilde{P}A = BP$. Our original system has a solution if and only if there is a solution to the equivalent diagonal system, which is of the form $\forall j \in \{0, ..., r\}, \widetilde{P_j}(b_j) = \widetilde{x_j}$. Since all nonzero endomorphisms in $R[\delta]$ are surjective, it suffices (to prove the existence of a solution) to prove that the diagonal coefficients $\widetilde{P_j}$ are all nonzero. Assume that $\widetilde{P_k} = 0$ for some $k \in \{0, ..., r\}$, and let e_k be the line vector whose coordinates are all zero except the k-th coordinate which is 1. Then, $\widetilde{P_k} = 0$ rewrites as $e_k\tilde{P} = 0$, and thus $e_kBP = e_k\tilde{P}A = 0$. Since B is invertible and $e_k \neq 0$, we know that the line vector e_kB is nonzero.

We shall prove that this is contradictory by showing that for every line vector $x \in R[\delta]^{r+1}$, the equality $xP = 0$ implies $x = 0$. Assume that $xP = 0$ for some nonzero vector $x \in R[\delta]^{r+1}$, i.e.

$$\forall j, \sum_{i=0}^{r} x_i P_{i,j} = 0.$$

Choose an index $m \in \{0, ..., r\}$ such that $\theta(x_m) \geq \theta(x_i)$ for all i. We have:

$$\sum_{i \neq m} x_i P_{i,m} = -x_m P_{m,m}$$

and therefore:

$$\begin{aligned}\theta(x_m) &\leq \max_{i \neq m}\left(\theta(x_i) + \theta(P_{i,m})\right) - \theta(P_{m,m}) \\ &< \theta(x_m) + \theta(P_{m,m}) - \theta(P_{m,m}) \\ &= \theta(x_m)\end{aligned}$$

which is a contradiction. This concludes the proof. □

Proof of Theorem 3.3.1. The necessary condition in Theorem 3.3.1 follows from Proposition 3.2.1. Let (R, δ) be a differential field such that every endomorphism in $R[\delta]$ is surjective. Choose a nonzero element $x \in R[\delta]$ and write it as:

$$x = \sum_{i=0}^{n} x_i \delta^i$$

with $x_n \neq 0$ (cf. Lemma 3.3.1.1). To prove that $R[\delta]$ is weakly fadelian, we want to find $b, c \in R[\delta]$ such that $bx + xc = 1$. It suffices for this to find coefficients $b_0, ..., b_{n-1}, c_0, ..., c_{n-1} \in R$ such that:

$$\left(\sum_{i=0}^{n-1} b_i \delta^i\right)\left(\sum_{i=0}^{n} x_i \delta^i\right) + \left(\sum_{i=0}^{n} x_i \delta^i\right)\left(\sum_{i=0}^{n-1} c_i \delta^i\right) = 1.$$

Rewrite this as:

$$\begin{aligned} 1 &= \sum_{i=0}^{n-1}\sum_{j=0}^{n}\left[\left(\sum_{k=0}^{i} b_i \binom{i}{k} x_j^{(k)} \delta^{i-k+j}\right) + \left(\sum_{k=0}^{j} x_j \binom{j}{k} c_i^{(k)} \delta^{j-k+i}\right)\right] \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n}\sum_{k=0}^{n}\left(b_i \binom{i}{k} x_j^{(k)} + x_j \binom{j}{k} c_i^{(k)}\right)\delta^{i+j-k} \end{aligned}$$

By Lemma 3.3.1.1, the coefficients in the decomposition of 1 are unique. So, we get a system of equations: for each $d \in \{0, ..., 2n-1\}$ (playing the role of $i + j - k$) we must solve:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n} b_i \binom{i}{i+j-d} x_j^{(i+j-d)} + x_j \binom{j}{i+j-d} c_i^{(i+j-d)} = \delta_{d,0}. \tag{2}$$

For $d = 2n - 1$, we get $b_{n-1}x_n + x_n c_{n-1} = 0$. Thus, we can express b_{n-1} as a function of c_{n-1}. Similarly, letting $d = 2n - 2$ lets one express b_{n-2} as a function of (the derivatives of) c_{n-2} and c_{n-1}, and so on until $d = n$. This process yields elements $A_{i,j} \in R[\delta]$ such that:

$$b_i = \sum_{j=i}^{n-1} A_{i,j}(c_j) \tag{3}$$

and such that $\theta(A_{i,j}) \leq n - i - 1 \leq n - 1$. Substitute b_i for $\sum_j A_{i,j}(c_j)$ in Equation 2 for $0 \leq d \leq n - 1$ to obtain a system of n equations of the form:

$$\sum_{i=0}^{n-1} P_{i,d}(c_i) = \delta_{d,0}. \tag{4}$$

The expression of b_i in Equation 3 involves only derivatives of $c_i, c_{i+1}, ..., c_{n-1}$ up to the $(n-1)$-th derivative. Thus, the only n-th derivatives that may appear in $P_{i,d}$ are in the terms:

$$x_j \binom{j}{i+j-d} c_i^{(i+j-d)}.$$

If $d < i$, the binomial coefficient is zero. If $d > i$, the derivative is of order $i + j - d < n$. If $d = i$, the term obtained for $j = n$ is $x_n c_d^{(n)}$ which effectively involves an n-th derivative with the nonzero coefficient x_n. This shows:

$$\theta(P_{d,d}) = n \qquad \text{and} \qquad \theta(P_{i,d}) \leq n - 1 \text{ for } i \neq d.$$

Hence the system given by Equation 4 is diagonally dominant and thus admits a solution by Lemma 3.3.2.1. This proves that $R[\delta]$ is weakly fadelian. □

Remark 3.3.2.2. We have another sufficient condition that does not require R to be commutative, but requires that all possible nonconstant polynomial differential equations have a solution in R, and not only linear ones. These are equations that may look something like:

$$a\left(X^{(19)}\right)^2 bX' - cX'dXeX^{(3)} = 0.$$

We do not include the proof here since this has not yielded new examples.

3.4. A non-Noetherian fadelian ring

We prove the following theorem:

Theorem 3.4.1. There exists a non-Noetherian[3] fadelian ring.

Proof. The key idea is that fadelianity is a first order property, and thus is preserved by ultrapowers, whereas Noetherianity is not. Let (k, δ) be a differentially closed field as above, with $\delta \neq 0$. Then $k[\delta]$ is fadelian by Theorem 3.3.1. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. The following ring is fadelian by Łoś's theorem [Mar02, Exercise 2.5.18] :

$$k[\delta]^{\mathcal{U}} = \frac{k[\delta]^{\mathbb{N}}}{\mathcal{U}}$$

Let $e_n \in k[\delta]^{\mathcal{U}}$ be the coset of:

$$\widehat{e_n} = \left(\underbrace{1_{k[\delta]}, 1_{k[\delta]}, ..., 1_{k[\delta]}}_{n}, \delta, \delta^2, \delta^3, ...\right) \in k[\delta]^{\mathbb{N}}.$$

Let I_n be the left ideal of $k[\delta]^{\mathcal{U}}$ generated by e_n. We have:

$$\widehat{e_{n-1}} = \left(\underbrace{1_{k[\delta]}, 1_{k[\delta]}, ..., 1_{k[\delta]}}_{n-1}, \delta, \delta, \delta, ...\right)\widehat{e_n}.$$

So $e_{n+1} \in I_n$, and the sequence of left ideals (I_n) is nondecreasing. We prove that it is strictly increasing by contradiction. Assume $e_n \in I_{n-1}$ for some $n \geq 0$. Then there is an element $a \in k[\delta]^{\mathbb{N}}$ such that:

$$\widehat{e_n} \sim a\widehat{e_{n-1}} = (a_1, ..., a_{n-1}, a_n\delta, a_{n+1}\delta^2, ...).$$

Consider the map $\psi : k[\delta] \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ defined in the following way: $\psi(0) = +\infty$, and otherwise write $x = \sum a_i \delta^i$ as in Lemma 3.3.1.1 and let $\psi(x) = \min\{i \mid a_i \neq 0\}$. For an element $\hat{x} \in k[\delta]^{\mathbb{N}}$, denote by $\psi(\hat{x})$ the element of $\left(\mathbb{Z}_{\geq 0} \cup \{+\infty\}\right)^{\mathbb{N}}$ obtained by evaluating ψ coordinatewise. Using $\geq$ to notate coordinatewise inequality, we have:

[3] By non-Noetherian, we systematically mean "neither left nor right Noetherian".

$$\psi(a\widehat{e_{n-1}}) = (\psi(a_0), ..., \psi(a_{n-1}), \psi(a_n) + 1, \psi(a_{n+1}) + 2, ...)$$
$$\geq \left(\underbrace{0, ..., 0}_{n-1}, 1, 2, 3, 4, ... \right).$$

On the other hand:

$$\psi(\widehat{e_n}) = \left(\underbrace{0, ..., 0}_{n-1}, 0, 1, 2, 3, ... \right).$$

Therefore $\widehat{e_n}$ and $a\widehat{e_{n-1}}$ may only have finitely many common coefficients, which contradicts the equivalence $\widehat{e_n} \sim a\widehat{e_{n-1}}$. So (I_n) is a strictly increasing sequence of left ideals. This contradicts left Noetherianity. We prove similarly that $k[\delta]^{\mathcal{U}}$ is not right Noetherian. □

3.5. A countable non-Noetherian fadelian ring

We prove Theorem 3.5.2, which states that there exists a countable non-Noetherian fadelian ring. First, we prove the following lemma:

Lemma 3.5.1. Let R be a fadelian ring and S be a subset of R. Let κ be the cardinal $\max(\aleph_0, |S|)$. There is a fadelian subring of R of cardinality $\leq \kappa$ which contains S.

Proof. We construct a sequence R_i of subrings of R of cardinality $\leq \kappa$ in the following way:
- R_0 is the subring of R generated by S;
- Assume we have constructed R_n. For every couple $x, a \in R_n$ such that $a \neq 0$, choose elements $b_n(x, a)$ and $c_n(x, a)$ in R such that $x = ab_n(x, a) + c_n(x, a)a$, using the fact that R is fadelian. Let R_{n+1} be the subring of R generated by R_n and the elements $b_n(x, a)$, $c_n(x, a)$ for all pairs $x, a \in R_n$ with $a \neq 0$.

Finally, define:

$$R_\infty = \bigcup_{n \geq 0} R_n.$$

Since R_∞ is the increasing union of a countable family of rings of cardinality $\leq \kappa$ containing S, it is itself a ring of cardinality $\leq \kappa$ containing S. To prove that R_∞ is fadelian, consider elements $x, a \in R_\infty$ with $a \neq 0$. There exists $n \in \mathbb{N}$ such that both x and a are in R_n. In R_{n+1} and therefore in R_∞, we have $x = ab_n(x, a) + c_n(x, a)a$. This proves that R_∞ is fadelian. □

Theorem 3.5.2. There exists a countable non-Noetherian fadelian ring.

Proof. Start with the non-Noetherian fadelian ring $R = k[\delta]^{\mathcal{U}}$ obtained in the proof of Theorem 3.4.1. Let S be the countable subset $\{u, e_0, e_1, e_2, ...\}$ of R, where u is the coset of $(\delta, \delta, \delta, ...) \in k[\delta]^{\mathbb{N}}$ and e_n is the coset of:

$$\widehat{e_n} = \left(\underbrace{1_{k[\delta]}, ..., 1_{k[\delta]}}_{n}, \delta, \delta^2, \delta^3, ...\right) \in k[\delta]^{\mathbb{N}}.$$

By Lemma 3.5.1, there is a countable fadelian subring R_∞ of R containing S. Using elements of S, we replicate the proof of Theorem 3.4.1 in R_∞: the sequence $(R_\infty e_n)_{n\geq 0}$ of left ideals of R_∞ is strictly increasing, and similarly for the right ideals $(e_n R_\infty)_{n\geq 0}$. We conclude that R_∞ is a countable non-Noetherian fadelian ring. □

4. Formal Laurent series on fadelian rings

In this section, we study formal series over a domain R. We define them in the following way:

Definition 4.1. The ring $R[[X]]$ is the ring of formal power series with coefficients in R where the indeterminate X commutes with elements of R, i.e. multiplication is given by:

$$\left(\sum_{n\geq 0} a_n X^n\right)\left(\sum_{n\geq 0} b_n X^n\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right) X^n.$$

If $P = \sum_{n\geq 0} a_n X^n$ is an element of $R[[X]]$, we denote by $P(0)$ the element $a_0 \in R$. We also define Laurent series over R:

Definition 4.2. The ring $R((X))$ consists of elements which are either 0 or of the form $X^j P$, where $j \in \mathbb{Z}$, $P \in R[[X]]$ and $P(0) \neq 0$, equipped with the product $(X^j P)(X^k Q) = X^{j+k}(PQ)$.

This construction is related to fadelianity because it preserves it:

Theorem 4.3. The domain R is fadelian if and only if $R((X))$ is fadelian.

Proof.

- First assume that $R((X))$ is fadelian. Let $x, a \in R \setminus \{0\}$. There are $X^j P, X^k Q \in R((X))$, with $P(0), Q(0) \neq 0$, such that:

  $$x = X^j P a + a X^k Q.$$

  By multiplying by some X^r, one may assume that r, j, k are three nonnegative integers, one of them zero, such that:

  $$x X^r = X^j P a + a X^k Q.$$

  If $r = 0$, then $x = X^j P a + a X^k Q$ in $R[[X]]$ and by evaluating at 0, we get $x \in Ra + aR$. Otherwise, we have $r \geq 1$ and either j or k is zero. We assume for example that $j = 0$. Then:

  $$x X^r = P a + a X^k Q.$$

Since $r \geq 1$, we know that $(xX^r)(0) = 0$. Moreover $(Pa)(0) = P(0)a$ is nonzero because R is a domain. Hence $\left(aX^kQ\right)(0)$ is also nonzero. This means that necessarily $k = 0$. We have the equality:

$$xX^r = Pa + aQ.$$

By evaluating this equality at 0, we get: $0 = P(0)a + aQ(0)$. Hence:

$$xX^r = (P - P(0))a + a(Q - Q(0)).$$

Both $P - P(0)$ and $Q - Q(0)$ cancel at 0. We can factor X from the equality:

$$xX^{r-1} = P_1 a + aQ_1.$$

Iterate the process to reach:

$$x = P_r a + aQ_r.$$

Finally, evaluate at zero to obtain the desired equality in R:

$$x = P_r(0)a + aQ_r(0).$$

This proves that R is fadelian.

- Now assume that R is fadelian. Consider two nonzero elements of $R((X))$ written as $X^jP,\ X^kQ$ with $j, k \in \mathbb{Z},\ P, Q \in R[[X]]$ and $Q(0) \neq 0$. Write $P = \sum_{n \geq 0} p_n X^n$ and $Q = \sum_{n \geq 0} q_n X^n$. Then q_0 is a nonzero element of R.

  We are searching for sequences $(b_0, b_1, b_2, ...)$ and $(c_0, c_1, c_2, ...)$ of elements of R such that:

$$\sum_{n \geq 0} p_n X^n = \left(\sum_{n \geq 0} q_n X^n\right)\left(\sum_{n \geq 0} b_n X^n\right) + \left(\sum_{n \geq 0} c_n X^n\right)\left(\sum_{n \geq 0} q_n X^n\right). \tag{5}$$

  If we find such sequences, then we have the equality in $R((X))$:

$$X^jP = \left(X^kQ\right)\left(\sum_{n \geq 0} b_n X^{n+j-k}\right) + \left(\sum_{n \geq 0} c_n X^{n+j-k}\right)\left(X^kQ\right)$$

  which shows that $R((X))$ is fadelian.

  We prove the existence of (b_n) and (c_n) by induction:

  - Looking at the constant coefficient in Equation 5, we get the equality:

$$p_0 = q_0 b_0 + c_0 q_0.$$

    We can fix $b_0, c_0 \in R$ satisfying this equality, because R is fadelian and $q_0 \neq 0$.

  - Assume we have defined $b_0, ..., b_{n-1}, c_0, ..., c_{n-1}$ such that the coefficients in front of X^i are equal in both sides of Equation 5, for $i = 0, ..., n-1$. Now consider the coefficient in front of X^n. We are trying to solve the equation:

$$p_n = \sum_{i=0}^{n} q_i b_{n-i} + c_{n-i} q_i$$

which can be rewritten as:

$$p_n - \sum_{i=1}^{n} (q_i b_{n-i} + c_{n-i} q_i) = q_0 b_n + c_n q_0.$$

We can fix $b_n, c_n \in R$ satisfying this equality, because R is fadelian and $q_0 \neq 0$.

□

To construct a non-Ore fadelian ring (Corollary 4.5), the main ingredient is the following theorem:

Theorem 4.4. Assume R is a countable fadelian ring which is not right (resp. left) Noetherian. Then $R((X))$ is a fadelian ring which is not right (resp. left) Ore.

A more general version holds (if R is a non-Noetherian countable simple domain, then $R((X))$ is not Ore) but we prove the weaker statement. Theorem 4.4 and Theorem 3.5.2 directly imply:

Corollary 4.5. There exists a fadelian ring which is neither right nor left Ore.

Proof of Theorem 4.4. We focus on right ideals, as the other case is dual. The ring $R((X))$ is fadelian by Theorem 4.3. Since R is countable and not right Noetherian, there are:

- a bijective enumeration $(a_0, a_1, a_2, ...)$ of all elements of R, with $a_0 = 0$;
- a strictly increasing sequence $0 = \widetilde{I_0} \subsetneq \widetilde{I_1} \subsetneq \widetilde{I_2} \subsetneq ...$ of right ideals of R.

Choose for every $n \geq 0$ an element $b_n \in \widetilde{I_{n+1}} \setminus \widetilde{I_n}$ and define the right ideal:

$$I_n = b_0 R + ... + b_n R \qquad \left(\subseteq \widetilde{I_{n+1}}\right).$$

Let $n \geq 1$. We shall prove that there is a $c_n \in R$ such that $b_n c_n a_n \notin I_{n-1}$. Assume by contradiction that there is no such c_n. Then $b_n R a_n \subseteq I_{n-1}$. Write $1 = \gamma_n a_n + a_n \delta_n$ for some $\gamma_n, \delta_n \in R$. Then:

$$b_n = b_n \cdot 1 = b_n(\gamma_n a_n + a_n \delta_n) = \underbrace{b_n \gamma_n a_n}_{\in b_n R a_n} + \underbrace{b_n \cdot 1 \cdot a_n}_{\in b_n R a_n} \delta_n \in I_{n-1} \subseteq \widetilde{I_n}.$$

This contradicts the choice of b_n as an element of $\widetilde{I_{n+1}} \setminus \widetilde{I_n}$. So we may choose a sequence $c_1, c_2, ...$ of elements of R such that $b_n c_n a_n \notin I_{n-1}$. We also define $c_0 = 1$.

Define the set $J_n = \{x \in R \mid b_n c_n x \in I_{n-1}\}$. For $n \geq 1$, the set J_n is a right ideal of R which does not contain a_n. Since the elements $(a_n)_{n \geq 1}$ form an exhaustive enumeration of $R \setminus \{0\}$, we have:

$$\bigcap_{n \geq 1} J_n = 0.$$

Now consider the two following nonzero elements of $R((X))$:

$$A = \sum_{n \geq 0} b_n c_n X^n \qquad \text{and} \qquad B = A + b_0.$$

To prove that $R((X))$ is not right Ore, it suffices to prove that $AT = BS$ implies $T = S = 0$. Assume by contradiction that $AT = BS$, where T and S are nonzero elements which we write as:

$$T = \sum k_n X^n \qquad \text{and} \qquad S = \sum l_n X^n.$$

By multiplying by some power of X, we may assume that $k_n = l_n = 0$ for negative n and that either $k_0 \neq 0$ or $l_0 \neq 0$. Look at the constant coefficient in the equality $AT = BS$ to obtain:

$$b_0 k_0 = 2 b_0 l_0.$$

Since R is a domain and $b_0 \neq 0$, we have $k_0 = 2l_0$. Now look at the coefficient in front of X^i in the equality $AT = BS$:

$$b_i c_i k_0 + \sum_{j=0}^{i-1} b_j c_j k_{i-j} = b_i c_i l_0 + b_0 l_i + \sum_{j=0}^{i-1} b_j c_j l_{i-j}.$$

Substitute k_0 by $2l_0$ in this equality and isolate the term $b_i c_i l_0$ to obtain:

$$b_i c_i l_0 = b_0 l_i + \sum_{j=0}^{i-1} b_j c_j \big(l_{i-j} - k_{i-j}\big).$$

Hence, $b_i c_i l_0$ belongs to the ideal I_{i-1}. This means that l_0 belongs to J_i, for all $i \geq 1$. As we have shown, $\bigcap_{i \geq 1} J_i = 0$. This implies $l_0 = k_0 = 0$, which is a contradiction. $\square$

Remark 4.6. The constructions of this article use the axiom of choice, e.g. in order to have a differentially closed field to apply Corollary 3.3.2 to, and in order to obtain a non-principal ultrafilter in the proof of Theorem 3.4.1. However, Maxime Ramzi has observed that Corollary 4.5 holds in ZF by the following argument: if V is a model of ZF and L is its constructible universe, then L is a model of ZFC and thus it proves that the first-order theory of non-Ore fadelian rings has a model. That model, seen in V, is also a model of the same first-order theory (we omit the verifications). By completeness, a choice-free proof of Corollary 4.5 exists.

Funnily, this argument does not work for Theorem 3.4.1, because the theory of non-Noetherian rings is not first-order. But since Corollary 4.5 implies Theorem 3.4.1, it is still true that Theorem 3.4.1 holds in ZF.

Another consequence of the argument is that the answer to the open question of whether there is a non-fadelian weakly fadelian ring does not depend on the axiom of choice.

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5. Formalizations of results from Section 2 in Lean 4.2

```
import Mathlib.Algebra.Ring.Basic
import Mathlib.Algebra.Ring.Defs
import Mathlib.Tactic.NthRewrite
import Mathlib.Tactic.NoncommRing
import Mathlib.Tactic.Existsi

-- Lean 4.2 version (2023-11-27)
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-- Fadelian, weakly fadelian rings
class Fadelian (R : Type*) [Ring R] [Nontrivial R]: Prop :=
  (prop : ∀(x:R), ∀(a:R), (a ≠ 0) → (∃(b:R), ∃(c:R), x = a*b + c*a))
class WeakFadelian (R : Type*) [Ring R] [Nontrivial R]: Prop :=
  (prop : ∀(a:R), (a ≠ 0) → (∃(b:R), ∃(c:R), 1 = a*b+c*a))

-- Fadelian rings are weakly fadelian
instance Fadelian.toWeakFadelian {R : Type*} [Ring R] [Nontrivial R] [Fadelian
R] : WeakFadelian R :=
  ⟨Fadelian.prop 1⟩

-- Left Ore rings
class LeftOre (R : Type*) [Ring R] [IsDomain R]: Prop :=
  (prop : ∀(x:R), ∀(y:R), (x≠0) → (y≠0) → ∃(a:R),∃(b:R),(a≠0) ∧ (b≠0) ∧ (a*x=b*y))

-- In a weakly fadelian ring, xy=yx=0 ⇒ x²=0 or y²=0
lemma lem_domain_1 {R :Type*} [Ring R] [Nontrivial R][WeakFadelian R] (x:R) (y:R)
(xy_zero : x*y=0) (yx_zero : y*x=0) : (x*x=0) ∨ (y*y=0) := by
  cases (em (x=0))
  case inl x_zero =>
    left
    rw [x_zero, mul_zero]
  case inr x_nonzero =>
    right
    obtain ⟨b,c,d⟩ := WeakFadelian.prop x x_nonzero
    nth_rewrite 1 [← mul_one y, d]
    simp [mul_add, ← mul_assoc, yx_zero]
    simp [mul_assoc, xy_zero]

-- In a weakly fadelian ring, x²=0 ⇒ x=0
lemma lem_domain_2 {R :Type*} [Ring R] [Nontrivial R] [WeakFadelian R] (x : R)
(xx_zero : x*x=0) : x=0 := by
  by_contra x_nonzero
  obtain ⟨b, c, d⟩ := WeakFadelian.prop x x_nonzero

  have cx_eq_cxcx : c*x=c*(x*c)*x := by
    rw [← mul_one (c*x), d, mul_add]
    simp [mul_assoc, xx_zero]
    simp [← mul_assoc, xx_zero]
  have xb_eq_xbxb : x*b=x*(b*x)*b := by
    rw [← one_mul (x*b), d, add_mul]
    simp [mul_assoc, xx_zero]
    simp [← mul_assoc, xx_zero]

  have xbxb_or_cxcx_zero : ((x*b)*(x*b)=0) ∨ ((c*x)*(c*x))=0 := by
    have cxxb_zero : (c*x)*(x*b) = 0 := by
```

```
      simp [← mul_assoc, mul_assoc, xx_zero]
    have xbcx_zero : (x*b)*(c*x) = 0
      := by
      have xb_from_cx : x*b = 1 - c * x := by simp [d]
     have xb_cx_commute : (x*b)*(c*x) = (c*x)*(x*b) := by rw [xb_from_cx, sub_mul,
one_mul, mul_sub, mul_one]
      rw [xb_cx_commute]
      simp [mul_assoc, ← mul_assoc, xx_zero]
    exact lem_domain_1 (x*b) (c*x) xbcx_zero cxxb_zero

  have one_eq_zero : ((0:R)=(1:R)) := by
    cases xbxb_or_cxcx_zero
    case inl xbxb_zero =>
      have xb_zero : x*b=0 := by
        rw [xb_eq_xbxb, mul_assoc, mul_assoc, ← mul_assoc, xbxb_zero]
      rw [xb_zero, zero_add] at d
      have ccxx_one : (c*c)*(x*x) = 1 := by
        rw [← mul_assoc, mul_assoc c c x, ← d, mul_one, ← d]
      rw [xx_zero, mul_zero] at ccxx_one
      apply ccxx_one
    case inr cxcx_zero =>
      have cx_zero : c*x=0 := by
        rw [cx_eq_cxcx, mul_assoc, mul_assoc, ← mul_assoc, cxcx_zero]
      rw [cx_zero, add_zero] at d
      have xxbb_one : (x*x)*(b*b) = 1 := by
        rw [← mul_assoc, mul_assoc x x b, ← d, mul_one, ← d]
      rw [xx_zero, zero_mul] at xxbb_one
      apply xxbb_one

  have x_zero : (x=0) := by
    rw [← one_mul x, ← one_eq_zero, zero_mul]
  exact x_nonzero x_zero

-- Weakly fadelian rings are domains
instance WeakFadelian.to_isDomain {R :Type*} [Ring R] [Nontrivial R] [WeakFadelian
R] : IsDomain R := by
  have : NoZeroDivisors R :=
  ⟨fun {x y xy_zero} => by
    have yx_zero : y*x=0 := by
      apply lem_domain_2 (y*x)
      simp [mul_assoc, ←mul_assoc, xy_zero]
    cases (lem_domain_1 x y xy_zero yx_zero)
    case inl xx_zero =>
      left
      exact lem_domain_2 x xx_zero
    case inr yy_zero =>
      right
      exact lem_domain_2 y yy_zero
  ⟩
  apply NoZeroDivisors.to_isDomain

-- Weakly fadelian left Ore rings are fadelian
theorem left_ore_weak_fadelian_is_fadelian {R :Type*} [Ring R] [Nontrivial R]
[WeakFadelian R] [LeftOre R] : Fadelian R :=
  ⟨fun {x a a_nonzero} => by
    cases (em (x=0))
    case inl x_zero =>
      existsi (0:R), (0:R)
```

```
    simp [x_zero]
  case inr x_nonzero =>
    obtain ⟨b, c, b_nonzero, _, bx_eq_ca⟩ :=
      LeftOre.prop x a x_nonzero a_nonzero
    have ab_nonzero : (a*b ≠ 0) := by
      simp [a_nonzero, b_nonzero]
    obtain ⟨k, l, abk_p_lab_eq_one⟩ :=
      WeakFadelian.prop (a*b) ab_nonzero
    existsi (b*k*x), (l*a*c)
    simp [mul_assoc, ←bx_eq_ca]
    simp [← mul_assoc, ← add_mul]
    simp [mul_assoc l a b, ← abk_p_lab_eq_one]
⟩
```